Today (June 3)

text reference: Ch 6, 7, 8, 9

- sets, set operations
- quantifers
- functions
- graphs
- properties

$\in \quad \forall \quad \exists \qquad \cap \quad \cup$

$\{ \ \} \quad | \quad \times \quad \setminus \quad {}^c$

$\ldots \qquad \emptyset$

Pseudo-definition: A [redacted] is a collection of things with no other information (no relations, no ordering, just the things themselves).

Ex: - The people in a classroom
- The atoms in a rabbit
- The odd numbers
- The pianos manufactured in countries with currently presiding female presidents

Like everything in mathematics, sets are often denoted by letters.
CAPITAL letters look larger, while ordinary lowercase letters are small.
Sets are "larger" than the things they contain (called elements), so a set might be called

$$A \quad \text{or} \quad B \quad \text{or} \quad \mathbb{Z} \quad \text{or} \quad \mathbb{C}$$

while an element might be called

a or b or 5 or i

$\alpha\ \beta\ \gamma\ \delta\ \underbrace{\varepsilon\ \epsilon}_{}\ \phi\ \iota\ \kappa\ \lambda\ \mu\ \nu\ \theta\ \omega\ \pi\ \rho\ \sigma\ \tau\ \chi\ \zeta$

η ?

$A\ B\ \Gamma\ \Delta\quad \Phi\ I\ K\ \Lambda\quad \Theta\ \Omega\ \Pi\ P\ \Sigma\ T$

functions

also numbers or things (integers?)

a b c d e f g h i j k l m n o p q r s t u v w x y z

numbers or things

indices

not really used

time

real numbers, spatial coordinates

$a \in A$ means a is the name for a specific element contained in the set called A

Briefly,

- a is contained in A
- A contains a
- a is in A.

$A = \{\square, \Delta, 5, 7, q\}$ means A is the set containing

$\square$, Δ, 5, 7, and q

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$$

$$A = \left\{ \text{letters or formulas/expressions} \;\middle|\; \begin{array}{c}\text{conditions involving the letters,} \\ \text{i.e. predicates}\end{array} \right\}$$

means

A is the set of all thing described by the letters or expressions which satisfy the conditions.

$$\text{Odds} = \{ n \in \mathbb{Z} \mid 2 \nmid n \}$$

$$\text{PMICWCPFP} = \left\{ \pi \;\middle|\; \begin{array}{l}\pi \text{ is a piano and} \\ \pi \text{ was manufactured in a country } C \text{ such that} \\ \text{the current president of } C \text{ is } P \text{ such that} \\ P \text{ is a female}\end{array} \right\}$$

$$R = \left\{ x \in \mathbb{R} \;\middle|\; \begin{array}{l}\text{For each integer } n\text{, all possible digits occur} \\ \text{with equal frequency in the } n\text{-ary representation} \\ \text{of } x\end{array} \right\}$$

"random numbers"

fact The "total length" of the set R in $\mathbb{R}$ is everything; the complementary set $\mathbb{R} \setminus R$ has zero length.

fact No examples of elements of R are known.

$A \subset B$ ("A is a subset of B")

means

$a \in A \Rightarrow a \in B$

$A = B$ ("A equals B")

means

$A \subset B$ and $B \subset A$

* Occasionally the symbol $\subseteq$ is used. ("or equal to")
You might then wonder, does $\subset$ mean "contained in but not equal to"?
No. (Ok, depends on the author, but not really)

$A \cap B$ the set of (all the things in A and also in B)

$A \cup B$ the set of (all the things in A) and (all the things in B)

$A - B$ or $A \setminus B$ the set all things in A which are not in B

$$A^c = \left(\begin{matrix}\text{all the things} \\ \text{in your scope}\end{matrix}\right) \setminus A$$

$$P(A) = \{ B \mid B \subset A \}$$

$$= \{ B \subset A \}$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

associativity

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

commutativity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

distributivity

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

De Morgan laws

When "for all ..." / "for every..." is used to delineate scope, it can be notated:

$$\forall$$

When "there exists a ...." is used to formulate an existential claim, it can be notated:

$$\exists$$

Mathematical notation allows you to make extreme abbreviations of definitions and claims.

For example, some of the things from last class in notation:

<u>def</u> $\forall a, b \in \mathbb{Z} \quad a \mid b \Leftrightarrow (\exists c \quad b = ac)$

prop $\forall a, b, c \in \mathbb{Z} \quad a \mid b \, , \, b \mid c \Rightarrow a \mid c$

prop $\nexists m, n \in \mathbb{Z} \quad 14m + 20n = 101$

Also,

<u>def</u> $\text{Odds} = \{ a \in \mathbb{Z} \mid \forall b \in \mathbb{Z} \quad 2b \neq a \}$

Exercise

For each item, choose meanings for the sets A and B and the statement P, and decide if the resulting statement is true.

1) $\forall a \in A \ \exists b \in B \quad P$

$A = \{\text{driving cars}\}$

$B = \{\text{humans}\}$

$P = b$ is driving a

False

2) $\exists a \in A \ \forall b \in B \quad P$

3) $\forall a \in A \ \forall b \in B \quad P$

4) $\exists a \in A \ \exists b \in B \quad P$

Product of sets

$A \times B$ means the set of all pairs (something from A, something from B).

$$A \times B = \{ (a, b) \mid a \in A, \ b \in B \}$$

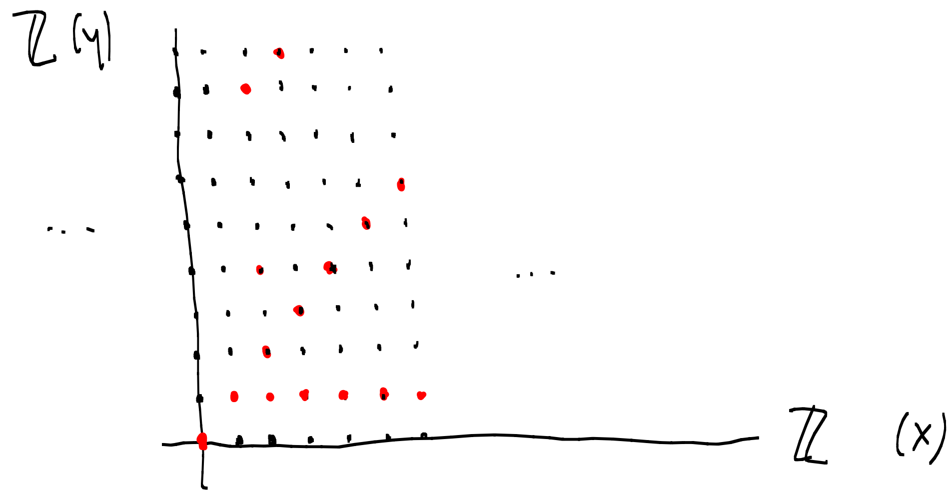
It's "obvious" that these pairs are ordered.

The product of a set with itself also is made of ordered pairs.

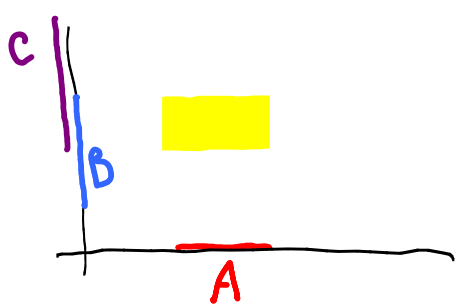
$$A \times A = \{ (a_1, a_2) \mid a_1, a_2 \in A \}$$

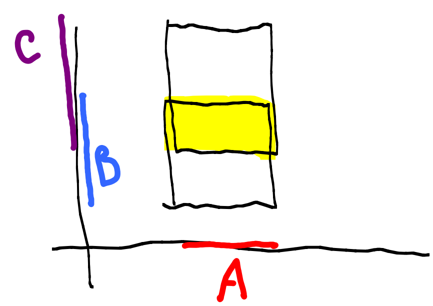
That is, for example, $(5,7)$ and $(7,5)$ are different in $\mathbb{Z} \times \mathbb{Z}$.

$$S = \{(x,y)\in \mathbb{Z}\times\mathbb{Z} \mid \exists n\in\mathbb{Z}\ \ n>0,\ y=x^n\}$$



$$A\times(B\cap C) = (A\times B)\cap(A\times C)$$





$$A\times B \cap C\times D = A\cap C \times B\cap D$$

Proof.

$$(x,y)\in A\times B\cap C\times D \iff x\in A,\ y\in B,\ x\in C,\ y\in D$$
$$\iff x\in A\cap C,\ y\in B\cap D$$
$$\iff (x,y)\in (A\cap C)\times(B\cap D)$$

Functions

<u>Pseudo-definition</u>: A function is an assignment of an output to certain inputs.

assignment
list of values
map
mapping
rule
relation

Notation:

$$f : X \to Y$$

$$x \mapsto f(x)$$

input

$$\text{image } f = \{ y \in Y \mid \exists x \in X \quad f(x) = y \}$$

<u>definition using sets</u> A function f from X to Y is a subset

$$G \subset X \times Y$$

such that

$$\left(\forall x \in X \ \exists y \in Y \quad (x, y) \in G \right)$$

and $\left(\forall x \in X \ \nexists y_1, y_2 \in Y \quad y_1 \neq y_2 \text{ and } (x, y_1), (x, y_2) \in G \right)$

G is called the graph of f.

- Given $f: X \to Y$, $A \subset X$

  $f|_A$ denotes restriction of f to A

- Given $f: X \to Y$, $g: Y \to Z$,

  $g \circ f$ denotes the composition

Exercise Let $G(f)$ and $G(g)$ be the graphs of f, g.
Write the graph $G(g \circ f)$ in set notation.

$$G(g \circ f) = \left\{ (x, z) \in X \times Z \;\middle|\; \exists y \in Y \quad \begin{array}{l} (x, y) \in G(f) \text{ and} \\ (y, z) \in G(g) \end{array} \right\}$$

identity function $\quad I_X : X \to X$

injective — one-to-one

surjective — onto

bijective — one-to-one and onto

For $f: X \to Y$,

f is injective means $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or $\{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\} = \phi$

f is surjective means [redacted]

or image $f = Y$

f is bijective means [redacted]

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f^{-1}

Suppose $f: X \to Y$. An inverse for f is a function $g: Y \to X$ such that

$f \circ g$ is equal to $I_Y : Y \to Y$

and $g \circ f$ is equal to $I_X : X \to X$

<u>theorem</u> There exists an inverse for f if and only if f is bijective.

<u>Next time:</u> Cardinality, inclusion/exclusion, sets of functions, permutations, pidgeonhole principle

Read Ch 12, 14