Today June 5

- cardinality
- finite and infinite sets
- inclusion/exclusion principle
- sets of functions
- permutations

HW #2

(1) For sets A, B, C, find a bijection

$$\mathrm{Fun}(A \times B, C) \to \mathrm{Fun}(A, \mathrm{Fun}(B, C))$$

(2) Problems I #26

(3) Read and understand the proof that the rational numbers $\mathbb{Q}$ are countable.
Wait 1 hour, then write a proof of this fact.
Then try to find a complete proof with pictures.

(4) Problems II #6, 7, 17

The technical term for the size of a set is cardinality.

Only pretty small sets have a size which is a number:

$$|\{1, 3, 5, 7\}| = 4$$

$$|\{0, 1, 2, \ldots 100\}| = 101$$

"$\mathbb{N}_n$"

<u>def</u> For sets A and B,

means

there exists a bijection $A \to B$ (an exact 1-1 correspondence).

Prop For sets X having the same cardinality as one of the sets

$$\mathbb{N}_n = \{1, 2, \dots, n\},$$

the definition "X has cardinality n" makes sense; that is,

If there are bijections $X \to \mathbb{N}_n$ and $X \to \mathbb{N}_m$ then $n = m$.

Proof. The point is that if this result were not true, we might have, for example, a situation where $X \xrightarrow{\cong} \{1,2,3\}$ and $X \xrightarrow{\cong} \{1,2,3,4,5\}$ (both bijections). Then we would be wondering is $|X| = 3$ or 5?

Proofs like this are all about notation:

Let $f: X \to \mathbb{N}_n$ and $g: X \to \mathbb{N}_m$ be bijections.

Then $g \circ f^{-1}: \mathbb{N}_n \to \mathbb{N}_m$ is also a bijection, because of this:

> **Lemma** Let $f: A \to B$ and $g: B \to C$ be functions.
>
> (1) If f and g are injective then so is $g \circ f$.
>
> (2) If f and g are surjective then so is $g \circ f$.

We will also prove the following, which will point to the pidgeonhole principle:

> **Lemma** If there exists an injection $\mathbb{N}_n \to \mathbb{N}_m$ then $n \leq m$.

Assuming this, we proceed:

Since $g \circ f^{-1} : \mathbb{N}_n \to \mathbb{N}_m$ is bijective, it is injective. Then $n \leq m$.

Similarly $f \circ g^{-1} : \mathbb{N}_m \to \mathbb{N}_n$ is bijective, hence injective, so $m \leq n$.

By the trichotomy law for real numbers,

$$(n < m \text{ or } n = m) \text{ and } (m < n \text{ or } m = n)$$

$$\Leftrightarrow (n < m \text{ and } m < n) \text{ or } (n = m \text{ and } m < n) \text{ or } (n < m \text{ and } m = n) \text{ or } (n = m \text{ or } m = n)$$

$$\Rightarrow n = m$$

The end.

Proof of that lemma

Formally, the statement is

$$\forall m \in \mathbb{Z}_+ \quad \forall n \in \mathbb{Z}_+ \left(\exists \text{ injection } \mathbb{N}_n \to \mathbb{N}_m \Rightarrow n \leq m \right)$$

Induction on m.

$m = 1$ | If $f : \mathbb{N}_n \to \mathbb{N}_1$ is an injection, $f(1) = 1$ and $f(n) = 1 \Rightarrow n = 1$.

Then $n \leq m$.

Suppose that for some fixed m, for each injection $\mathbb{N}_n \to \mathbb{N}_m$, $n \leq m$.

Now pick an injection $f : \mathbb{N}_n \to \mathbb{N}_{m+1}$.

There are 2 cases:

Case 1: $\text{Image}(f) \not\ni m+1$

In this case there is an injection $\text{Image} f \to \mathbb{N}_m$ (called "the inclusion").

The composition of injections is a new injection:

$$\mathbb{N}_n \xrightarrow[\text{of codomain}]{\text{restriction}} \text{Image } f \longrightarrow \mathbb{N}_m$$

Then by induction $n \leq m$, so $n \leq m+1$ (addition law).

$$\boxed{\text{Case 2: Image } f \ni m+1}$$

In this case there is a $k \in \mathbb{N}_n$ such that $f(k) = m+1$.

Define a new function $g : \mathbb{N}_{n-1} \to \mathbb{N}_m$,

$$g(x) = \begin{cases} f(x) & \text{if } x < k \\ f(x+1) & \text{if } x \geq k \end{cases}$$

We can check directly that g is an injection:

Given $x_1, x_2 \in \mathbb{N}_n$,

	$x_1 < k$	$x_1 \geq k$
$x_2 < k$	$f(x_1) = f(x_2)$ $\Rightarrow x_1 = x_2$	same
$x_2 \geq k$	$f(x_1) = f(x_2+1)$ $\Rightarrow x_1 = x_2+1$ But $x_1 - x_2 \leq k-k=0$ CONTRADICTION	$f(x_1+1) = f(x_2+1)$ $\Rightarrow x_1+1 = x_2+1$ $\Rightarrow x_1 = x_2$

Then by induction $n-1 \leq m \Rightarrow n \leq m+1$

Theorem Pidgeonhole principle

If X and Y are finite sets and $|X| > |Y|$, then for any function $f: X \to Y$, there exist $x_1, x_2 \in X$ such that

$$f(x_1) = f(x_2)$$

Proof The contrapositive is, if there exists a function $f: X \to Y$ that is injective, then $|X| \leq |Y|$. This is the lemma we already proved.

Exercise: Prove that if X and Y are finite sets of the same size, then for any function $f: X \to Y$,

$$f \text{ is injective} \iff f \text{ is surjective}$$

Inclusion-exclusion Let X and Y be finite sets. Then

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$

$$\begin{aligned} \mathrm{Fun}(X,Y) &= \{ f : X \to Y \} \\ &= \{ G \subset X \times Y \mid \text{the conditions for } G \text{ to be the graph of a function hold} \} \end{aligned}$$

$$|\mathrm{Fun}(X,Y)| = |X|^{|Y|} \quad \text{if } |X|, |Y| \text{ are finite}$$

Exercise How many injections $X \to Y$ are there?

How many surjections $X \to Y$ are there? $\leftarrow$ hard...

A <u>permutation</u> is a bijection $X \to X$.

<u>Exercise</u> How many permutations of X are there?

<u>Exercise</u> What is $|\mathcal{P}(X)|$ (assuming X is finite)?

<u>Exercise</u> Prove that $\binom{n}{k} = \binom{n}{n-k}$. (Hint: the binomial coefficients are the sizes of some specific sets)

<u>Infinity</u>

defs A set X is called <u>finite</u> if there exists a bijection $X \to \mathbb{N}_n$ for some $n \in \mathbb{Z}$.

A set X is called <u>infinite</u> if it is not finite.

A set X is called <u>countable</u> if it is finite or there is a bijection $X \to \mathbb{Z}_+$.

A set X is called <u>uncountable</u> if it is not countable.

prop. If A, B are countable, so is $A \cup B$ and $A \times B$.

__theorem__ $\mathbb{Q}$ is countable. $\mathbb{R}$ is not.

General discussion of cardinalities and ordering.

$$|X| \qquad \text{(no real meaning)}$$

$$|X| \leq |Y|$$

$$|X| < |Y|$$

__Ex)__ $P(X), X$