Different proofs

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A proposition or theorem typically admits several different proofs. Often the same proof can even be expressed in many ways–verbal, visual, symbolic, or some mix–reflecting the diversity of purposes to which proofs are put. Are you writing to:

-Explain to a general audience?
-Convince others that the statement is true?
-Convince others (who already know why the statement is true) that you know why the statement is true?
-Decide for yourself if the statement is true?
-Decide what statement should be true?

Binomial theorem.
For any number x and any positive integer n,

$$(1+x)^n = \sum_{i=0}^{i=n} \binom{n}{i} x^i$$

Proof 1. Base case $n = 1$:

$$1 + x = \binom{1}{0} 1 + \binom{1}{1} x$$

Inductive step:

$$
\begin{aligned}
(1+x)^n &= \sum_{i=0}^{i=n} \binom{n}{i} x^i \\
\implies (1+x)^n(1+x) &= \left(\sum_{i=0}^{i=n} \binom{n}{i} x^i \right)(1+x) \\
\implies (1+x)^{n+1} &= \sum_{i=0}^{i=n} \binom{n}{i} x^i + \sum_{i=0}^{i=n} \binom{n}{i} x^{i+1} \\
\implies (1+x)^{n+1} &= \sum_{i=0}^{i=n} \binom{n}{i} x^i + \sum_{i=1}^{i=n+1} \binom{n}{i-1} x^i \\
\implies (1+x)^{n+1} &= \left(\sum_{i=1}^{i=n} \left(\binom{n}{i} + \binom{n}{i-1} \right) x^i \right) + \binom{n}{0} x^0 + \binom{n}{n} x^{n+1} \\
\implies (1+x)^{n+1} &= \left(\sum_{i=1}^{i=n} \binom{n+1}{i} x^i \right) + \binom{n+1}{0} x^0 + \binom{n+1}{n+1} x^{n+1} \\
\implies (1+x)^{n+1} &= \sum_{i=0}^{i=n+1} \binom{n+1}{i} x^i
\end{aligned}
$$

Discussion of Proof 1. This is the kind of proof you write for yourself when you're trying to decide if the statement is even true. Mechanical deductions, one after the other, make the reasoning plain. If there were an error, it would be easy to spot. The Pascal identity for the binomial coefficients and certain values of these coefficients are used in the proof without any special mention. It is bad practice but sometimes clearer for external results to be used without comment, even in proofs designed to be read by others, as long there is enough context.

Proof 2. We will use induction. The base case $n = 1$ is trivial.

Suppose that the statement is true for some $n > 0$. Then:

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) \\
&= \left(\sum_{i=0}^{i=n} \binom{n}{i} x^i\right)(1+x) \\
&= \left(\sum_{i=1}^{i=n} \left(\binom{n}{i} + \binom{n}{i-1}\right) x^i\right) + 1 + x^{n+1} \\
&= \sum_{i=0}^{i=n+1} \binom{n+1}{i} x^i
\end{aligned}$$

The last step follows from the Pascal identity:

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$$

Discussion of Proof 2. This is the same proof as Proof 1, made shorter and clearer. Some of the details are omitted and the main step–the application of the Pascal identity–is made explicit. This is important to include because the reader who doesn't know this extra result might waste time wondering if the step can be made with elementary arithmetic. On the other hand, the detailed steps that *can* be made with elementary observations can usually be safely omitted. For example, the fact that $\binom{n}{n} = \binom{n+1}{n+1} = 1$, used explicitly in Proof 1, may be considered obvious enough to anyone who knows the definition of the symbol $\binom{a}{b}$.

Proof 3. According to the distributivity laws, in the expansion of $(1+x)^n$ there will be one term for each simultaneous choice of term, 1 or x, from each of the n factors. The term corresponding to such a choice will be x^i, where i is the number of factors in which x was chosen instead of 1. According to the definition of the binomial coefficients, $\binom{n}{i}$ is precisely the number of ways to select i things out of n things. Then the number of ways to choose i special factors, out of all n factors, from which to select x (instead of 1) is $\binom{n}{i}$. Since each one of these contributes one x^i to the sum,

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + ... + \binom{n}{n-1} x^{n-1} + x^n$$

Discussion of Proof 3. This is a completely different, direct proof. It is harder to tell precisely what I mean, because the symbolic language is largely replaced by explanatory written English. But, when the meaning is understood, the main point is much clearer than in the case of the symbolic proofs.