MAT 200 sample exam questions

A. Non-technically, a *graph* is an abstract system of vertices connected by edges. Technically, a graph is: a set V whose elements are called *vertices*, and a subset $E \subset V \times V$ whose elements are called *edges*. An element $(v_1, v_2) \in E$ is said to *connect* v_1 to v_2. Two graphs (V, E) and (V', E') are called *equivalent* if there is a bijection $f : V \to V'$ such that $F(E) = E'$, where F is the bijection $V \times V \to V' \times V'$ defined by $(v_1, v_2) \mapsto (f(v_1), f(v_2))$.

1. Draw pictures of 3 examples of graphs with 5 vertices which are all inequivalent to each other, and also draw the subset E inside the set $V \times V$. (Use the usual "coordinate" picture for the Cartesian product.)

2. How many graphs can be made using 3 vertices? 4? How about n vertices, for an integer n?

3. Using this definition, how many edges can connect two given vertices?

4. Within this framework, define *loop*. In one of your graphs from part (1), mark the edges of a loop in the Cartesian product picture.

B. The set of symmetries of objects forms what is called a *group*. A *group* is a set G, a distinguished element called $1 \in G$ (or 0), and an operation $* : G \times G \to G$ called the multiplication (or addition), such that

1. (Inverses exist) For each $g \in G$, there exists an element h such that $g * h = 1$.

2. (Associativity) For each $g, h, k \in G$, $(g * h) * k = g * (h * k)$.

Consider the following.

1. Let P_n denote the set of permutations of our set $\mathbb{N}_n = \{1, 2, 3..., n\}$ (that is, bijections $\mathbb{N}_n \to \mathbb{N}_n$). Verify that P_n and the operation $*$ defined by

$$(f * g)(x) = f(g(x))$$

   form a group.

2. The notation $f = (235)(41)(87)$ means f is a permutation of $\mathbb{N}_8$ that maps 2 to 3, 3 to 5, 5 to 2, 4 to 1, 1 to 4, 8 to 7, and 7 to 8. Write the function f by specifying its values on the inputs $\mathbb{N}_n$ written in the usual order.

3. If $f = (a_1 a_4)(a_2 a_3)$ and $g = (a_3 a_2 a_4)$, what is $f(g(a_4))$? Can you make a table of all values of $f * g$? Can you represent $f * g$ in this "cycle" notation?

4. The *order* of a permutation f is the smallest integer n such that f^n is the identity function (Here f^2 means the function $f \circ f$, f^3 means $f \circ f \circ f$..., and the identity function is the function that maps each x to x). What is the order of the permutation $f * g$ found in part (3)?

5. The rigid motions of the plane form a group called the *Euclidean isometry group*, using the operation of function composition. Technically, a "rigid motion" is a bijection $f : P \to P$ from the set of points P of the plane to itself such that the distance between any two $a, b \in P$ is the same as the distance between $f(a)$ and $f(b)$. Can you classify the elements of order 2 in this group? How about the elements with infinite order (no finite number is the order of the element)?

C. In calculus you work with functions $f : \mathbb{R} \to \mathbb{R}$. For a given input $x \in \mathbb{R}$, f is called *continuous at x* if

For every number $b > 0$, there exists a number $a > 0$ such that the condition $|f(y) - f(x)| < b$ holds when $|y - x| < a$. In other terms:

$$\forall b > 0 \, \exists \, a > 0 \, \forall y \, (|y - x| < a \implies |f(y) - f(x)| < b)$$

1. Write the negation (logical opposite) of "f is continuous at x" using the quantifer notation.

2. Find a specific function f which is not continuous at $x = 1$. Prove that it is not continuous there. (This hardly needs to be said: *from the definition*)

3. Assume a given function f is continuous at $x = 1$. Prove that the function g defined by $g(x) = f(x) \cdot f(x)$ is also continuous at 1.

4. Using (3), decide whether the function $f(x) = x^2$ is continuous at $x = 1$.

D. Consider a hemisphere and a tangent plane in 3-dimensional space. We may regard the hemisphere as the set H:

$$H = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} | x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}$$

and the plane as the set P:

$$P = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} | z = 1\}$$

Define a function p from the 3-space to the plane P by the formula $p(x, y, z) = (x/z, y/z, 1)$. (Actually, p is not defined at $(0, 0, 0)$.) p is called a *central projection*. It can be defined without formulas: $p(X)$ is the intersection of the line through 0 and X with the plane P.

1. Let p_H be the projection from 3-space to H defined by: $p_H(X)$ is the intersection of the line through 0 and X with H. Let p_{HP} be the projection from H to P defined by: $p_{HP}(X)$ is the intersection of the line through 0 and X with the plane P. Write formulas for p_H and p_{HP}.

2. Using the formulas, prove that $p_{HP} \circ p_H = p$.

3. Without using the formulas, prove that $p_{HP} \circ p_H = p$.

4. Verify that the formula $h(x, y, 1) = \left(\frac{x}{\sqrt{1+x^2+y^2}}, \frac{y}{\sqrt{1+x^2+y^2}}, \frac{1}{\sqrt{1+x^2+y^2}} \right)$ defines a function whose image is contained in H, and that this function defines an inverse for p_{HP}. Denote it by p_{PH} (for "projection from P to H"); conclude that p_{HP} is a bijection.

5. For a given angle θ, the formula

$$f(x, y, z) = (\cos(\theta)x + \sin(\theta)z, y, -\sin(\theta)x + \cos(\theta)z)$$

defines a function from the 3-space to itself called the *rotation about the y-axis by angle θ*. Compute a formula for $p \circ f$ as a function from P to P (actually, some points are missing from the domain: Why? It is better to regard P as some of the points of the projective plane). This function is called a *perspectivity.* What is the interpretation of $p \circ f$ in terms of visual perspective?