MAT 200 sample exam solutions C and D

C. In calculus you work with functions $f : \mathbb{R} \to \mathbb{R}$. For a given input $x \in \mathbb{R}$, f is called *continuous at* x if

For every number $b > 0$, there exists a number $a > 0$ such that the condition $|f(y) - f(x)| < b$ holds when $|y - x| < a$. In other terms:

$$\forall b > 0 \, \exists \, a > 0 \, \forall y \, (|y - x| < a \implies |f(y) - f(x)| < b)$$

1. Write the negation (logical opposite) of "f is continuous at x" using the quantifer notation.

2. Find a specific function f which is not continuous at $x = 1$. Prove that it is not continuous there. (This hardly needs to be said: *from the definition*)

3. Assume a given function f is continuous at $x = 1$. Prove that the function g defined by $g(x) = f(x) \cdot f(x)$ is also continuous at 1.

4. Using (3), decide whether the function $f(x) = x^2$ is continuous at $x = 1$.

Solution.
1. There exists a number $b > 0$ such that for all numbers $a > 0$ there exists a number y such that $|f(y) - f(x)| > b$ even though $|y - x| < a$. In other terms,

$$\exists b > 0 \, \forall a > 0 \, \exists y \, (|f(y) - f(x)| > b \text{ and } |y - x| < a)$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by the rules $f(x) = 0$ if $x \leq 1$ and $f(x) = 2$ if $x > 1$. f is discontinuous at $x = 1$: For the number $b = 1$, for any positive number a, there exists a number y, for example $y = 1 + \frac{a}{2}$, such that $|f(y) - f(x)| > b$ and $|y - x| < a$. These conclusions hold since:

$$\begin{aligned} &|f(y) - f(x)| = |2 - 0| = 2, \\ &b = 1, \\ &2 > 1 \end{aligned}$$

and

$$\begin{aligned} &|y - x| = |1 + \frac{a}{2} - 1| = \frac{a}{2}, \\ &\frac{a}{2} < a \end{aligned}$$

3. Assume f is continuous at $x = 1$. Pick $b > 0$, and set $b' = \min\{\frac{b}{3|f(1)|}, |f(1)|\}$ so that $b' < \frac{b}{3|f(1)|}$ and $b' < |f(1)|$. By continuity of f at $x = 1$, we may assume there exists $a' > 0$ such that for all y, $|y-1| < a' \implies |f(y) - f(1)| < b'$. Then set $a = a'$. If $|y-1| < a$, then $|y-1| < a'$, so $|f(y) - f(1)| < b'$, and:

$$\begin{aligned}
|g(y) - g(1)| &= |f(y)^2 - f(1)^2| \\
&= |f(y) - f(1)| \cdot |f(y) + f(1)| \\
&= |f(y) - f(1)| \cdot |(f(y) - f(1)) + 2f(1)| \\
&\leq b' \cdot (b' + 2|f(1)|) \\
&\leq \left(\frac{b}{3|f(1)|}\right) \cdot (|f(1)| + 2|f(1)|) \\
&= b \\
&\implies \\
|g(y) - g(1)| &< b
\end{aligned}$$

Therefore g is continuous at $x = 1$.

4. The function $g(x) = x^2$ is equal to the function defined by $f(x) \cdot f(x)$ defined in (3), where $f(x) = x$. Since the function f is continuous (choose $a = b$ in the definition), so is g (by 3!).

D. Consider a hemisphere and a tangent plane in 3-dimensional space. We may regard the hemisphere as the set H:

$$H = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} | x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}$$

and the plane as the set P:

$$P = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} | z = 1\}$$

Define a function p from the 3-space to the plane P by the formula $p(x, y, z) = (x/z, y/z, 1)$. (Actually, p is not defined at $(0, 0, 0)$.) p is called a *central projection.* It can be defined without formulas: $p(X)$ is the intersection of the line through 0 and X with the plane P.

1. Let p_H be the projection from 3-space to H defined by: $p_H(X)$ is the intersection of the line through 0 and X with H. Let p_{HP} be the projection from H to P defined by: $p_{HP}(X)$ is the intersection of the line through 0 and X with the plane P. Write formulas for p_H and p_{HP}.

2. Using the formulas, prove that $p_{HP} \circ p_H = p$.

3. Without using the formulas, prove that $p_{HP} \circ p_H = p$.

4. Verify that the formula $h(x, y, 1) = \left(\frac{x}{\sqrt{1+x^2+y^2}}, \frac{y}{\sqrt{1+x^2+y^2}}, \frac{1}{\sqrt{1+x^2+y^2}}\right)$ defines a function whose image is contained in H, and that this function defines an inverse for p_{HP}. Denote it by p_{PH} (for "projection from P to H"); conclude that p_{HP} is a bijection.

5. For a given angle θ, the formula

$$f(x, y, z) = (\cos(\theta)x + \sin(\theta)z, y, -\sin(\theta)x + \cos(\theta)z)$$

defines a function from the 3-space to itself called the *rotation about the y-axis by angle θ*. Compute a formula for $p \circ f$ as a function from P to P (actually, some points are missing from the domain: Why? It is better to regard P as some of the points of the projective plane). This function is called a *perspectivity.* What is the interpretation of $p \circ f$ in terms of visual perspective?

Solution.

1. Given $X = (x, y, z)$ in 3-space, we must find the scaled version which has length equal to 1. We can simply divide it by its length:

$$p_H(x, y, z) = \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Given $X = (x, y, z)$ in H, we must find the scaled version which has z coordinate equal to 1. We can simply divide it by its z coordinate:

$$p_{HP}(x, y, z) = \left(\frac{x}{z}, \frac{y}{z}, 1\right)$$

2.

$$p_{HP} \circ p_H(x, y, z) = p_{HP}(p_H(x, y, z)) = p_{HP}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$= \left(\frac{\frac{x}{\sqrt{x^2+y^2+z^2}}}{\frac{z}{\sqrt{x^2+y^2+z^2}}}, \frac{\frac{y}{\sqrt{x^2+y^2+z^2}}}{\frac{z}{\sqrt{x^2+y^2+z^2}}}, 1\right) = (x/z, y/z, 1)$$

The formulas for $p_{HP} \circ p_H$ and for p are the same.

3. Pick X in 3-space. By definition, $p_H(X)$ is the intersection of the line through 0 and X with H. By definition, $p_{HP}(p_H(X))$ is the intersection of the line through 0 and $p_H(X)$ with the plane P. Since this line equals the original line through 0 and

X, $p_{HP}(p_H(X))$ is also equal to the intersection of the line through 0 and X with the plane P. This is also, by definition, $p(X)$.

4. Let $X = (x, y, z) = \left(\frac{a}{\sqrt{1+a^2+b^2}}, \frac{b}{\sqrt{1+a^2+b^2}}, \frac{1}{\sqrt{1+a^2+b^2}}\right)$ be an arbitrary point in the image of h (in this case, $X = h(a, b)$). X is in H because

$$x^2 + y^2 + z^2 = \frac{a^2}{1 + a^2 + b^2} + \frac{b^2}{1 + a^2 + b^2} + \frac{1}{1 + a^2 + b^2} = 1$$

and

$$\frac{1}{1 + a^2 + b^2} > 0$$

h is an inverse for p_{HP}; for $X = (x, y, z)$ belonging to H,

$$h(p_{HP}(x, y, z)) = h(\frac{x}{z}, \frac{y}{z}, 1) = \left(\frac{x/z}{\sqrt{1 + x^2/z^2 + y^2/z^2}}, \frac{y/z}{\sqrt{1 + x^2/z^2 + y^2/z^2}}, \frac{1}{\sqrt{1 + x^2/z^2 + y^2/z^2}}\right)$$
$$= (x, y, z)$$

and for $X = (x, y, 1)$ belonging to P,

$$p_{HP}(h(x, y, 1)) = p_{HP}\left(\frac{x}{\sqrt{1 + x^2 + y^2}}, \frac{y}{\sqrt{1 + x^2 + y^2}}, \frac{1}{\sqrt{1 + x^2 + y^2}}\right)$$
$$= \left(\frac{\frac{x}{\sqrt{1+x^2+y^2}}}{\frac{1}{\sqrt{1+x^2+y^2}}}, \frac{\frac{y}{\sqrt{1+x^2+y^2}}}{\frac{1}{\sqrt{1+x^2+y^2}}}, 1\right) = (x, y, 1)$$

5. For $X = (x, y, 1)$ belonging to P,

$$p(f(x, y, 1)) = \left(\frac{\cos(\theta)x + \sin(\theta)}{-\sin(\theta)x + \cos(\theta)}, \frac{y}{-\sin(\theta)x + \cos(\theta)}, 1\right)$$

$p \circ f$ is the transformation of the viewing plane P effected by rotating your head by angle θ around the y-axis.