Change of basis

There are 2 standard ways to record a change of basis in $\mathbb{R}^n$ from an old basis v_i to a new one w_i:

1. The linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ that sends each v_i to w_i, and its matrix in the v_i basis:

$$P = [L]_{v_i} = \left[\begin{pmatrix} \\ w_1 \\ \text{in v basis} \end{pmatrix} \begin{pmatrix} \\ w_2 \\ \text{in v basis} \end{pmatrix} \ldots \right]$$

   So $L(v_i) = w_i$

2. A matrix that converts column vectors expressed in the v_i basis into their expression in the w_i basis:

$$B = \left[\begin{pmatrix} \\ v_1 \\ \text{in w basis} \end{pmatrix} \begin{pmatrix} \\ v_2 \\ \text{in w basis} \end{pmatrix} \ldots \right]$$

   So $B \cdot [a]_{v_i} = [a]_{w_i}$

Actually, $B = P^{-1}$ and $P = B^{-1}$, so while B converts expressions in the old basis into their expressions in the new, P converts the new to the old.

things that can be represented	by matrices (in basis v_i)	change of basis formulas	type of tensor
linear maps $L : \mathbb{R}^n \to \mathbb{R}^n$	$a_{ij} = (L(v_j))_i$	$M' = BMB^{-1} = P^{-1}MP$	$\mathbb{R}^n \otimes \mathbb{R}^{n*}$
bilinear forms $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$	$a_{ij} = L(v_i, v_j)$	$M' = (B^{-1})^T MB^{-1} = P^T MP$	$\mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$
bilinear forms $L : \mathbb{R}^{n*} \times \mathbb{R}^{n*} \to \mathbb{R}$	$a_{ij} = L(v_i^*, v_j^*)$	$M' = BMB^T = P^{-1}M(P^{-1})^T$	$\mathbb{R}^n \otimes \mathbb{R}^n$

- With the matrix M of a linear map L with respect to v_i, $M \cdot [a \text{ in } v_i \text{ basis}] = [L(a) \text{ in } v_i \text{ basis}]$, and

$$BMB^{-1}\,[a \text{ in new } w_i \text{ basis}] = [L(a) \text{ in new } w_i \text{ basis}]$$

- With the matrix M of a bilinear form L with respect to v_i, $L(a, b) = [a \text{ in } v_i \text{ basis}]^T\, M\, [b \text{ in } v_i \text{ basis}]$, and also

$$L(a, b) = [a \text{ in new } w_i \text{ basis}]^T\, (B^{-1})^T MB^{-1}\, [b \text{ in new } w_i \text{ basis}]$$