MAT 211 (Summer I) Linear Algebra Final Exam

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Name:

Don't write decimals. Work out the solutions on separate pages before writing them in the space provided.

1. Consider the vector space V whose elements are the infinitesimal rotations of 3-dimensional space. That is,

$$V = \{3 \text{ by } 3 \text{ matrices } M \text{ such that } M^T = -M\}$$

Define an operation called the commutator or bracket $[,]$ that takes a pair $M, N \in V$ and produces the matrix

$$[M, N] = MN - NM$$

(i) Show that the bracket is bilinear and anti-symmetric.

(ii) Show that the matrix $[M, N]$ is always an element of V.

V can be considered $\mathbb{R}^3$ by regarding the vector (a_1, a_2, a_3) as the vector (matrix) in V

$$\begin{bmatrix} 0 & a_3 & a_2 \\ -a_3 & 0 & a_1 \\ -a_2 & -a_1 & 0 \end{bmatrix}$$

(iv) Show that if M corresponds to (a_1, a_2, a_3) and N corresponds to (b_1, b_2, b_3), then $[M, N]$ corresponds to the vector cross product:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

(v) So the cross product is the bracket. Use the bracket to demonstrate the vector identity

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0 \text{ for all } a, b, c \in \mathbb{R}^3$$

2. The solutions of differential equations appearing in practical applications in engineering and the sciences are often approximated by the method of finite elements. Central to this method is the ability to approximate the directional derivatives of functions whose values are only known on some triangular mesh (rectangular meshes are much easier). Suppose a real-valued function $f(x,y)$ is known at some triangular grid points, for example $(-1,-1)$, $(0,1)$, and $(3,0)$ as shown:

$$
\begin{aligned}
f(-1,-1) &= 1 \\
f(0,1) &= 3 \\
f(3,0) &= 5
\end{aligned}
$$

Our best guess for the graph of f in $\mathbb{R}^3$ is the 2-dimensional plane through the points

$$
\begin{aligned}
(-1,-1,1) \\
(0,1,3) \\
(3,0,5)
\end{aligned}
$$

(i) All of these points can be translated so that one of them lies at the origin $(0,0,0)$. Write the translated versions $a, b \in \mathbb{R}^3$ of the other two.

(ii) a and b span a 2-plane. It is the graph of a linear function of x and y that approximates f. Write a and b as a matrix of **rows**.

(iii) Perform Gaussian elimination on this matrix.

(iv) Since the row space hasn't changed, the rows of the resulting matrix span the same 2-plane. Draw these two vectors on a diagram in $\mathbb{R}^3$ and lightly fill in the plane they span.

(v) Use (iv) to estimate the directional derivatives of f in the x and y directions in the vicinity of this triangle. That is, find how much f changes when you change x by 1 and when you change y by 1.

3. There are two serfs and one king. Every day the king takes $\frac{1}{2}$ of each serf's money and gives each of them only $\frac{1}{12}$ of his.

(i) If x is the amount of money the king has and y and z are the amounts of money the serfs have, what are the new x, y, and z after one day?

(ii) Write the effect of a day's business as a matrix M applied to the column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(iii) Verify that $\lambda = \frac{1}{2}, \frac{1}{3}, 1$ are the eigenvalues of this matrix.

(iv) The eigenvectors with eigenvalues $\frac{1}{2}$ and $\frac{1}{3}$ have some negative entries, so they aren't that meaningful (our model isn't designed to account for negative money, debt). Compute the eigenvector with eigenvalue 1.

(v) Suppose that everyone starts out with the same amount of money, 8000 Mongolian tugrik. In the long run how much money will the king have? And the serfs?

4. Choose an n by n matrix M.

(i) Compute the singular values of M (the eigenvalues of $M^T M$).

(ii) Compute the length-squared of M as a vector in $\mathbb{R}^{n^2}$.

(iii) Why is the sum of the singular values the same as the length-squared of M?

(iv) Show that changing M into QM for some orthogonal matrix Q does not change the matrix $M^T M$.

(v) You should expect this (iv) even before doing the calculation. Why? (Recall that $M^T M$ is the matrix of the dot product viewed from the reference frame of the column vectors of M.)

(vi) Show that M and MQ have the same length (even though $M^T M \neq (MQ)^T MQ$).

5. Recall that a square matrix M satisfies its own characteristic equation (Cayley-Hamilton theorem). Suppose that M is a 2 by 2 matrix with trace t and determinant t^2.

(i) What is the characteristic polynomial of M?

(ii) Write M^2 as a linear combination of M and the identity matrix I.

(iii) Write M^3 as a linear combination of M and the identity matrix I.

(iv) What are the eigenvalues of M^3?

(v) What are the eigenvectors of M^3?