Actions, orbits, and Burnside's theorem

- The rotation group of the icosahedron has 60 elements, and it is transitive on vertices, meaning that any vertex can be moved to any other by an appropriate symmetry. There are 5 edges, arranged symmetrically, connected to any given vertex. How many vertices are there?
- The rotation group of a tetrahedron has 12 elements (it is the alternating group A_4.) There are 4 vertices. What is the smallest rotation about a vertex appearing as a symmetry of the tetrahedron?

Consider a rotation R of the cube of order 3, for example the element that we were representing last week as (123)(546) (the faces of a standard die are numbered 1 through 6).

- List all the elements of $\langle R \rangle$, the subgroup generated by R, and give an isomorphism $\langle R \rangle \to \mathbb{Z}_3$.
- This $G = \mathbb{Z}_3$ acts on the set of 8 vertices X. Draw a picture of this set, grouped by orbit. The orbit of a point x under the action of G is the set $\{g \cdot x | g \in G\}$ of all points you can get x to by applying symmetries.
- On a graph with the horizontal axis X and the vertical axis $G = \mathbb{Z}_3$, circle all grid points (x, g) such that $g \cdot x = x$.
- For each $x \in X$, determine $|G_x|$.
- For each $g \in G$, determine $|X^g|$.
- Compute $\sum_{x \in X} |G_x|/|G|$
- Compute $\sum_{g \in G} |X^g|/|G|$
- How many orbits are there?