Envelopes, with an application to visual surface geometry

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Abstract

The basic general theory of envelopes of families of topological spaces and smooth manifolds is presented, including notions of rank, order, and directions of stationarity, as well as the relation to contact and multi-contact geometry.

An algebraic lemma is proved that elegantly solves the problem of plane intersections and spans in a linear space. It works even in the cases of arrangements in special position. The formula turns out to have the auxiliary function of calculating the edge of regression of a developable surface in projective 3-space (and, presumably, envelopes of plane families in higher dimensions). For this reason the formula is considered from several points of view, including the general linear invariant theory of exterior, or skew-symmetric tensors, via the umbral calculus.

Finally, an application is given to visual surface reconstruction.

Contents

1 Introduction

In section 2.1 we define a notion of the envelope of a family of subspaces of a topological space. This notion is sufficient to describe the envelopes appearing in a surprisingly large number of contexts, particularly when the base of the family is a 1-dimensional manifold.
However, deeper study requires the differentiable setting introduced in section 2.2. A notion of stationary directions appears as soon as the dimension of the space parameterizing the family is larger than 1, since a family may envelop in some directions but not in others, and at some places rather than others. This leads also to envelope rank (or nullity), as a measure of the number of envelopable directions. The more subtle envelope order, introduced in section 35, measures something like density. Section 35 also contains a brief proposal for a recursive notion of jet-order contact between submanifolds of an ambient manifold.
In certain cases there is a handy algebro-geometric reformulation. It is partially developed in section 2.3.
The contactization described in section 2.4 is most familiar for 1-parameter families of hypersurfaces, in which case the theory presented here transmutes into the theory of caustics and contact geometry à la Arnold [Arn89].
In section 3.1 we make use of the structure maps of the exterior bialgebra ΛV to calculate the intersections and spans of linear subspaces of a projective space with respect to Plücker line and plane coordinates. We focus on the case of line intersections in 3 dimensions, because it applies to the calculation of the edge of regression of a developable surface and the focal surfaces of a

line congruence. The Grosshans-Rota-Stein [GRS87] extension of the umbral calculus of general linear invariants is brought to bear in order to obtain explicit formulas. The details are presented partly because they provide an example of an interesting combinatorial structure, somewhat reminiscent of a root system (see Figure 3).

The last section, 3.3.3, presents the corollary application to visual surface reconstruction which was the original motivation for this paper. Curiously, a “half-geodesic” condition, appearing here as a characterization of certain envelopable directions in the line families which arise from the visual setup, turns out to be a key ingredient in the projective-plane-rolling construction used by Bor, Lamoneda, and Nurowski to construct the exceptional Lie algebra $\mathfrak{g}_2$ [BLN15].

2 Envelopes of topological and smooth families

2.1 Topological setting

2.1.1 Families, presentations, and envelopes

By a *space* we always mean a Hausdorff topological space. Unless indicated otherwise, smooth means smooth of class C^∞.

Definition 1. *A* topological family of subspaces *of a space X is a subspace $E \subset B \times X$.*

We regard E as a family of subspaces of X parameterized by B, so that for example E_b means the image in X of the fiber of E over $b \in B$.

Definition 2. *An* implicit presentation *of a topological family $E \subset B \times X$ is a continuous function $F : B \times X \to \mathbb{R}^k$ whose zero set is E.*

Definition 3. *Two implicit presentations F and F' of a topological family are called* equivalent *if their zero-sets are equal, and their germs near the zero-set are conjugate by some homeomorphism of $\mathbb{R}^k$ relative to 0 and some homeomorphism of a neighborhood in $B \times X$ relative to the zero-set.*

Note that there are usually many inequivalent presentations of the same topological family. The primary purpose of this definition of implicit presentations of topological families will be to define the corresponding differential notion.

By the limit of a sequence of spaces in X we understand the set of limit points of all sequences of points drawn from the sequence of spaces.

Definition 4. *The b^{th}* big/small characteristic *of a topological family $E \subset B \times X$ is the union/intersection, over all sequences b_i converging to $b \in B$ with $b_i \neq b$, of the limit spaces of the sequence of intersections $E_b \cap E_{b_i} \subset X$.*

Definition 5. *The* big/small envelope *of a topological family $E \subset B \times X$ is the disjoint union of its big/small characteristics, endowed with the subspace topology it inherits from the inclusion into E.*

Informally we sometimes wish the envelope of a family to mean, more simply, the not-necessarily-disjoint union of the characteristics in X. That is, the image in X of the envelope as described above.

2.1.2 Examples

Example 6. Consider the 1-parameter family of lines in the plane near the top edge of a square, not meeting the interior of the square, and containing either the top right or top left vertex. Let b_0 denote the member containing the top edge. Then the big characteristic of the family at b_0 consists of the two top vertices, while the small characteristic of the family at b_0 is empty.

Example 7. Each non-degenerate differentiable plane curve is equal to both the big and small envelopes of its topological family of tangent lines. For every sequence of tangent lines $l_i \neq l$ converging to a fixed tangent line l at a point p, the intersections $\{l_i \cap l\}$ will form a collection of points whose limit set is $\{p\}$. Thus the intersection and the union over all such sequences of the limit set is $\{p\}$.

For degenerate plane curves containing a portion of a straight line, there are two things that we may mean by its family of tangent lines. Either the family contains repeated members for each point of the curve on the straight portion, or the family has only one member for each distinct tangent line.

In the repeated case, both the big envelope and the small envelope contain both the curve and the complete straight line extending the straight portion.

In the distinct-tangent-lines case, the small envelope consists of the open complement of the straight portion in the curve, and the big envelope consists of the closed complement of the straight portion in the curve.

Example 8. A non-degenerate submanifold of a linear or projective linear space of any dimension is equal to the small envelope of the topological family of all of its tangent *hyper*planes, also called its *dual*. Here non-degenerate can be taken to mean that its dual is a smooth hypersurface in the dual projective space.

These examples were partly intended to show that the notions of big and small envelopes are both needed.

Non-example 9. One might expect that a generic space curve is the big or small envelope of its topological family of tangent lines. With the given definitions, this is evidently false. Most nearby tangent lines will not intersect at all, so that both envelopes are empty.

This difficulty is insurmountable as long as we insist on a purely topological notion. One can construct a homeomorphism from a neighborhood of a generic space curve to $\mathbb{R}^3$ under which the tangent line family is straightened to a parallel family, and the image of the space curve bears no special relation to it. Thus there can be no way of ascertaining the curve from purely topological properties of its family of tangent lines.

2.2 Smooth setting

2.2.1 Families, presentations, and envelopes

Definition 10. *A* smooth family *or* family of submanifolds *of a smooth manifold X is a topological family $E \subset B \times X$ in which B is a smooth manifold and E is a smooth submanifold.*

We shall almost always assume that $\dim E > \dim B$ in order that the members $E_b \subset X$ of the family have positive expected dimension $\dim E - \dim B$, and that $E \to B$ is surjective, so that these members are not empty.

Definition 11. *A smooth family $E \subset B \times X$ is said to be:*

- high-dimensional *if* $\dim E > \dim X$, *or equivalently* $\dim B > \operatorname{codim} E$
- full *if* $\dim E = \dim X$ *or* $\dim B = \operatorname{codim} E$
- low-dimensional *if* $\dim E < \dim X$ *or* $\dim B < \operatorname{codim} E$.

The point is that in the low-dimensional and full cases, a reasonable picture of the total space E can be made in X; one expects discrete pre-images in E of points of X.

Definition 12. *An* implicit presentation *of a smooth family $E \subset B \times X$ is an implicit presentation $F : B \times X \to \mathbb{R}^k$ of the underlying topological family which is smooth and for which 0 is a regular value.*

Note 13. Since the pre-image of a regular value of a smooth map has codimension equal to the dimension of the target, an implicit presentation of a smooth family E of codimension k in $B \times X$ must have target $\mathbb{R}^k$. Also, although implicit presentations always exist locally, the existence of an implicit presentation as defined above implies that the normal bundle of E is trivial. Thus a smooth family may not admit a global such presentation.

Definition 14. *Two implicit presentations F, F' of a smooth family are called* equivalent *if they are equivalent as presentations of the underlying topological family, and the homeomorphisms appearing in the equivalence are diffeomorphisms.*

The following proposition justifies the use of presentations of describe smooth families.

Proposition 15. *Let $E \subset B \times X$ be a smooth family with compact closure. Any two implicit presentations F, F' of E are equivalent.*

Proof. This is really a fact about implicit presentations of submanifolds $E \subset M$ in the special case $M = B \times X$. Any two are germ conjugate.
We prove this mild generalization. Select a Riemannian metric on M. The exponential map restricts to a diffeomorphism from a neighborhood of the zero section of the normal bundle of E in M onto a neighborhood of E in M, which thereby acquires the structure of a fiber bundle N over E.
Consider the restrictions of F and F' to a small neighborhood of the zero section of this fiber bundle. By the regularity and compactness assumptions, if the neighborhood is sufficiently small, the maps F and F' are diffeomorphisms from each fiber onto its image in $\mathbb{R}^k$. The family of composite diffeomorphism germs $\mathbb{R}^k \to \mathbb{R}^k$, parameterized by E, given by $F \circ F'^{-1}$ (or $F' \circ F^{-1}$), conjugates F to F'. Said another way, $F^{-1} \circ F'$ (or $F'^{-1} \circ F$) restricts to a diffeomorphism germ $N \to N$ conjugating F to F'. $\square$

Thus, although equivalences between implicit presentations of smooth families are more discriminating than topological equivalences of underlying topological families—one expects more equivalence classes in the smooth case than in the topological one—the regular value condition on the objects comprising the classes in the smooth case is strong enough that there is at most one class.

Definition 16. *The* big/small envelope *of a smooth family $E \subset B \times X$ is the subset of E consisting of points at which the tangent space TE non-trivially intersects/contains the tangent space TB.*

Definition 17. *The* big/small characteristics *of a smooth family $E \subset B \times X$ are the members of the big/small envelope, regarded as a family parameterized by B.*

Remark 18. (Relation to the topological notion). Definitions 16 and 17 for a smooth family are sometimes compatible with Definitions 4 and 5 for the underlying topological family, especially when E is a 1-dimensional family of smooth hypersurfaces. However in many important cases they disagree, as described in Non-example 9.

Proposition 19. *The big envelope of a smooth family is the locus in E for which the rank of $E \to X$ is less than* $\dim E$.

Proof. TB is the kernel of the tangent map of the projection $B \times X \to X$. $TB \cap TE$ is the kernel of the tangent map of the projection $E \to X$. Thus a point $e \in E$ belongs this locus if and only if $T_eB \cap T_eE$ is non-zero. This is the defining condition for e to belong to the big envelope of the family E. □

Thus, although the way that the envelopes are organized into characteristics depends on the projection $E \to B$, the envelopes themselves are defined purely in terms of the differentiable map $E \to X$.

Proposition 20. *Let $F : B \times X \to \mathbb{R}^k$ be an implicit presentation of a smooth family $E \subset B \times X$.*
The big envelope is the locus of $e \in E$ such that $dF|_e$ has submaximal rank when restricted to TB.
The small envelope is the locus of $e \in E$ such that $dF|_e$ is zero when restricted to TB.

Proof. $e = (b, x) \in E$ belongs to the big envelope if and only if there is a non-zero vector $v \in T_bB \cap T_eE$. Along E, the kernel of dF is the tangent bundle of E, so the restriction of dF to TB is zero precisely on vectors of the form $v \in TB \cap TE$. This proves the first statement.
As for the second statement: $dF|_e$ is identically zero upon restriction to T_bB if and only if $\ker dF|_e = T_eE$ contains T_bB at e. By definition this is equivalent to the condition that e belongs to the small envelope. □

Corollary 21. *If F presents the smooth family E, the small envelope of E is locally presented by $(F, \partial_1 F, \partial_2 F, ...)$, where ∂_i are a local basis of vectors fields on B, lifted to $B \times X$.*

2.2.2 Envelope rank

Definition 22. *The* rank r (or nullity n) envelope *of a smooth family $E \subset B \times X$ is the locus of points in E at which the differential of the map $E \to X$ has rank less than or equal to r (or nullity greater than or equal to n).*

Definition 23. *The* rank r (or nullity n) characteristics *of a smooth family $E \subset B \times X$ are the members of the rank r (or nullity n) envelope, regarded as a family parameterized by B.*

When equality is desired in Definition 22, I will use the phrase "rank exactly r" or "nullity exactly n".

Proposition 24. *The big and small envelopes of a smooth family $E \subset B \times X$ are equal to the envelopes of nullity n and rank r with respective n and r given as follows:*

	n	r
low-dimensional big envelope	1	$\dim E - 1$
low-dimensional small envelope	$\dim B$	$\dim E_b$
full big envelope	1	$\dim E - 1$
full small envelope	$\dim B$	$\dim E_b$
high-dimensional big envelope	$\dim E - \dim X$	$\dim X$
high-dimensional small envelope	$\dim B$	$\dim E_b$

Note 25. If the family is high-dimensional, then the big envelope is always equal to all of E. Also, if B is 1-dimensional, the big and small envelopes are equal and there are no envelopes of different ranks.

Definition 26. *The* stationary vector bundle *on the envelope of nullity exactly n is the rank n vector bundle equal to the kernel of the restriction of the tangent map of $E \to X$ to the points of this envelope.*

Note 27. The stationary vector bundle of rank n is always tangent to E and defined precisely along the envelope of nullity exactly n, but it is not assumed to be tangent to this or any other envelope of E. On the other hand, it is everywhere tangent to the distribution TB in $B \times X$.

Definition 28. *For each $b \in B$, the projection of the stationary bundle of rank n into T_bB defines a family of $(n-1)$-dimensional projective-linear subspaces of $\mathbb{P}T_bB$, parameterized by the smooth locus of the b^{th} nullity n characteristic. It shall be called the* stationary family of $(n-1)$-planes at b.

See Example 45 in section 2.6 for an example of a stationary family.

2.2.3 Envelope order

The big envelope is stratified by rank, all the way down to the small envelope. We shall stratify each stratum even further. This refinement will make sense in the setting of any map $E \to X$ given as a composite map $E \subset M \to X$, where $M \to X$ is a submersion; that is, M need not necessarily be a product $B \times X$, as in our definition of a family $E \subset B \times X$.

Definition 29. *Suppose that submanifolds P and Q of an ambient manifold meet at a point x. Then P and Q are said to* make dimension n contact of order s at x *if there are n-dimensional submanifolds of P and of Q containing x whose s-jets at x are equal.*

Note 30. If either P or Q is one-dimensional, there is no need to specify the dimension n of the submanifolds appearing in Definition 29; it only makes sense to use $n = 1$.

Note 31. For a full treatment of jets, see [BCG+91] and [Olv86]. For the reader's convenience we recall briefly the notion which is used in Definition 29, the s-jet of a submanifold of dimension n of an ambient manifold A. The smooth maps $\mathbb{R}^n \to A$ are the smooth sections of the bundle $\mathbb{R}^n \times A \to \mathbb{R}^n$. The jet bundles of this bundle form a tower of finite-dimensional smooth fiber bundles $J^s \to A$, for $s = 0, 1, 2, \ldots$. Each of these bundles naturally receives a section corresponding to a given function $f : \mathbb{R}^n \to A$, called the s-jet lift or s^{th} derivative and denoted $j^s f$. The lift $j^s f$ is specified uniquely by the values of the derivatives of the component functions of f up to order s, when f is expressed with respect to coordinates on A.
The diffeomorphism group of $\mathbb{R}^n$ relative to a chosen basepoint in $\mathbb{R}^n$ acts on the fiber of this tower over the basepoint, and its orbit space in the fiber in J^s is what we are calling the space of *s-jets of submanifolds* of A.

Note 32. (*Recursive refinement*) The notion of order s contact between submanifolds P and Q can be restated as follows. For each dimension n, the spaces $J^{s,n}P$ and $J^{s,n}Q$ of s-jets of n-dimensional submanifolds of P and of Q are usually themselves smooth submanifolds P' and Q' of the manifold of s-jets of n-dimensional submanifolds of the ambient manifold. P and Q make dimension n contact of order s if these two spaces intersect.
So we are only considering whether or not P' and Q' have non-empty intersection. Evidently we may apply this definition recursively, to obtain a whole series of refined notions of contact between submanifolds. That is, we may consider the order of contact between P' and Q', the order of contact between iterated jet spaces P'' and Q'', and so on.
This would seem to be a very rich structure. It is certainly not obvious, for example, whether or not there are constraints imposed on the types of jet-contact which can arise at each level. Neverthless, this structure is present in the very minimal setting of local differential topology; to define it, no

additional data like metrics, tensor fields, vector bundles, or sheaves are needed. This recursive refinement is not used in this paper.

Example 33. Generic tangent lines to a surface make contact of order 1.

Note 34. Jet-contact differs slightly from point-contact in affine or projective geometry, or intersection multiplicity in algebraic geometry. There, an ordinary tangent line has 2-point contact, being the limit of secant lines through 2 points, while an asymptotic tangent line has 3-point contact, being the limit of secant lines through 3 points, etc. A line making $(s+1)$-point contact with a surface has jet-contact of order s with it.

Definition 35. *The* envelope of order s and nullity exactly n *of a smooth family $E \subset B \times X$ is the locus of points $e \in E$ at which the fiber of the projection $B \times X \to X$ (or $M \to X$ in general) makes exactly n-dimensional contact of order s with E at e.*

Note 36. (*Terminology*) Here "exactly" means that we exclude points making contact of dimension higher than n. Also, as defined, the envelopes of order s belong to each envelope of lower order. As in the case of rank, when we wish to discuss the points of the order s envelope which do not belong to any higher-order envelope, we shall use the phrase "envelope of order exactly s".

Proposition 37. *Suppose that $F : B \times X \to \mathbb{R}$ presents the smooth family E, so that E is a hypersurface, and suppose further that $\dim B = 1$, so that E is a smooth family of hypersurfaces. Its envelope of order s is equal to the zero set of the functions $F, \partial_t F, \partial_t^2 F, ..., \partial_t^s F$, where ∂_t is any non-vanishing vector field tangent to the fibers $B \times \{x\}$.*

Proof. This is a mild generalization of ([Eis09] page 59). □

Definition 38. *The* stationary bundle of order s and nullity exactly n *on the envelope of order s and nullity exactly n is the subbundle of $J^s(n, E)$ consisting of s-jets of n-dimensional submanifolds of E making contact with the fibers of the projection $B \times X \to X$.*

Note 39. The stationary bundle of order 1 and nullity exactly n is the projectivization of the stationary vector bundle of rank n of Definition 26.

2.3 Projective algebraic setting

Definition 40. *An* affinely-parameterized algebraic family of affine hypersurfaces *is a smooth family $E \subset B \times X$ in which B and X are affine spaces and E is a smooth algebraic hypersurface.*

One can substitute smaller-dimensional subvarieties but we do not consider them here.
Let $\mathbb{P}V$ denote the projective space with affine chart $B \times X$. Set E' equal to the projectivization of E in $\mathbb{P}V$. The projectivizations of the parallel affine subspaces $B \times \{x\}$ all meet in a projective plane B' at infinity of dimension $\dim B - 1$ in P.
Let X' denote the projective space of planes of $\mathbb{P}V$ of dimension $\dim B$ containing B'.
The analogue of the map $E \to X$ in this algebraic setting is the projection $E' \backslash B' \to X'$.
In the following proposition, we use the notation $\lrcorner$ to indicate contraction of a vector and a tensor, and the notation $\cdot$ to indicate symmetric product.

Proposition 41. *Suppose that* $\dim B = 1$*, so that* B' *is a point of* $\mathbb{P}V$ *and* X' *is the projective space of lines through* B'*. Let* $v \in V$ *denote a vector representing this point in* $\mathbb{P}V$*, and let* $F \in S^d V^\vee$ *represent the hypersurface* E'*.*
The envelope of the smooth family $E \subset B \times X$ *is equal to the affine part of the zero set* $Z(F, F \lrcorner v)$*, and the envelope of order* s *is given by the zero set* $Z(F, F \lrcorner v, F \lrcorner v \lrcorner v, ..., F \lrcorner (v \cdot ... \cdot v))$*, where* $v \cdot ... \cdot v$ *is the* s*-fold symmetric power of* v*.*

Proof. This follows from Proposition 37. See also [Dol12], page **. □

I do not know the replacement for this proposition when $\dim B > 1$, when the projection of the hypersurface takes place along planes emanating from a line or higher-dimensional plane, rather than along lines emanating from a point.
A naive construction fails in a curious way. In the case $\dim B = 2$, suppose that $v \wedge w \in \Lambda^2 V$ is the Plücker coordinate of the line in projective space toward which we will project. The contraction $F \lrcorner (v \wedge w)$ is evidently zero; it is a value of the square of the Koszul differential, $\Lambda^2 V \otimes S^d V^\vee \to V \otimes S^{d-1} V^\vee \to S^{d-2} V^\vee$.

2.4 Contactization

We observed that many of the features of a smooth family $E \subset B \times X$ are accessible by means of the map $E \to X$ given as the composite $E \subset M \to X$, where $M = B \times X$. This is one way to "unparameterize" the family, in such a way that the specification of the members cannot be recovered.

There is another way which implicitly remembers the parameter space B, and also embeds the enveloping question into the context of contact geometry.
For generic $e = (b, x) \in E$, the tangent space $T_x(E_b)$ has dimension $d = \dim E - \dim B$ and determines a point $p(e)$ of the fiber bundle $\mathrm{Gr}_d(TX)$.

Definition 42. *The topological space equal to the image of the assignment $e \mapsto p(e) \in \mathrm{Gr}_d(TX)$ is called the* contactization *of the smooth family $E \subset B \times X$. If a smooth family is desired, the* contactization *may mean further the set of smooth points of the closure of this space.*

The contactization C of a family is evidently foliated by d-dimensional manifolds, each covering one of the members $E_b \subset X$. These manifolds are maximally integral for the multi-contact structure of $\mathrm{Gr}_d(TX)$, and hence the E_b are recoverable from C.
Often $e \mapsto p(e)$ is a diffeomorphism $E \to C$, i.e. $E \to X$ factors through C. In this case the big envelope is identified with the locus of rank less than $\dim C$ of the projection $C \to X$. Moreover a notion of envelope order (perhaps different from Definition 35) can be defined by the order of contact between C and the fibers of the ambient projection $\mathrm{Gr}_d(TX) \to X$.
In the case that E is a 1-parameter family of hypersurfaces, the smooth contactization is a submanifold of the genuine contact manifold $\mathbb{P}T^*X$, and the integral foliation is by Legendre submanifolds. In this case the envelope is also known as the *caustic*. Caustics are defined as the singular locus of any composition, an inclusion of a Legendre submanifold in a contact manifold followed by a submersion with Legendre fibers [Arn89].

2.5 Iterated envelopes

The envelope of a family is a new family, whose members are the characteristics, which itself has an envelope, and so on.

Example 43. Consider a generic 1-parameter family of 2-planes in 3-space. Its envelope is a surface whose characteristic curves are lines. Such surfaces are called ruled. The ruled surfaces which arise in this way are known to be also *developable*, meaning that the envelope of the family of characteristic lines is a non-empty curve or a point, known as the edge of regression. Actually, in this example the first envelope and the second envelope obtained by iteration are precisely equal to the envelopes of order $s = 1$ and $s = 2$ of the original plane family (see [Eis09] pages 59-61).
Conversely, every developable surface is the envelope of its tangent planes. These tangent planes are constant along each line of the ruling, so although

this family of tangent planes would seem to be parameterized by a base of dimension 2, in fact this dimension is 1.

Note 44. Initially the base B of the family may not change during the process of iterated envelopes. However if $\dim B > 1$, eventually the dimension of the family will be too small to have non-empty fibers over every point of B. So in practice B may be replaced by successively smaller subspaces.

2.6 Universal families

Often one considers families whose members E_b belong to a natural finite-dimensional universal family of subspaces of X of a prescribed type. For example, the lines or planes of a linear space, the circles of a conformal geometry, or the geodesics of a homogeneous Riemannian geometry. The descent of enveloping properties of the universal family to sub-families is not trivial, but it is usually the best place to begin the study of a sub-family.

Example 45. Consider the 4-dimensional family B of all lines in the real projective 3-space (so $\dim E = 5$, $\dim X = 3$). The base B can be identified with the Grassmannian $\mathrm{Gr}(2, 4)$. Each line l is itself a characteristic of nullity exactly 2 of the family E. It turns out that its stationary family, parameterized by l, is a family of projective lines in $\mathbb{P}T_lB \cong \mathbb{P}^3$ constituting one ruling of a ruled quadric N_l. The points of this quadric are identified with the completions of l to a complete flag, by means of a point of l and a plane containing l. They are known as the *null directions* of B, at l. This structure plays a key role in twistor theory (see e.g. [WW90]).

3 Line and plane families

3.1 Linear intersections and spans

In this section a simple formula is presented which calculates the intersections and spans of any number of linear subspaces of a linear space. It has the character of a closed formula rather than an algorithm, making it conceptually simpler than the techniques of numerical linear algebra, like Gaussian elimination or matrix factorizations, usually applied to this situation.
We shall see in the next section, section 3.2, that it also calculates envelopes of families of lines in 3-space.
Consider a finite-dimensional vector space V, and its graded exterior algebra $\Lambda := \Lambda V$. Recall that there is a coproduct $\Delta : \Lambda \to \Lambda \otimes \Lambda$, the unique algebra homomorphism such that $\Delta(v) = 1 \otimes v + v \otimes 1$ for $v \in V$, where

$\Lambda \otimes \Lambda$ is endowed with the algebra structure of the graded-commutative tensor product. That is, for example, $(1 \otimes v) \cdot (v \otimes 1) = -(v \otimes 1) \cdot (1 \otimes v)$. The formula for $\Delta(e_1 \dots e_k)$ is a sum over all left/right groupings of $e_1, \dots, e_k$, keeping their order within each group, signed by the sign of the shuffle permutation achieving this ordered grouping.

The fact that Δ is an algebra homomorphism with respect to the wedge product $\wedge$ is the main condition that makes $(\Lambda, \wedge, \Delta)$ into a *bialgebra* in the graded sense. This fact is equivalent to equality of the maps depicted by the following pair of diagrams (read from top to bottom):

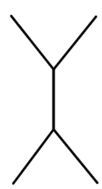


Figure 1: The coproduct algebra homomorphism condition is equality of the maps depicted by these two diagrams.

In fact, Λ is a *Hopf algebra* in the graded sense. Λ is also a *Frobenius algebra*, but *not* with the coproduct Δ; Hopf algebras are rarely also Frobenius. The Frobenius condition is expressed by equality of the maps depicted by the following three diagrams:

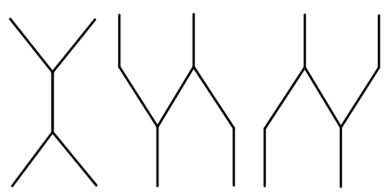
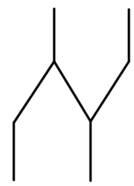


Figure 2: The Frobenius condition is equality of the maps depicted by these diagrams.

Since $(\Lambda, \wedge, \Delta)$ does not satisfy the Frobenius condition, the maps depicted in Figure 2 with respect to $\wedge$ and Δ are not equal. However, they are interesting:

Definition 46. *Define* $\psi^{p\,q}_{k\,(p+q-k)} : \Lambda^p \otimes \Lambda^q \to \Lambda^k \otimes \Lambda^{p+q-k}$ *to be the components of the composition of the coproduct, associator, and product indicated below:*

$$\psi : \Lambda \otimes \Lambda \to (\Lambda \otimes \Lambda) \otimes \Lambda \cong \Lambda \otimes (\Lambda \otimes \Lambda) \to \Lambda \otimes \Lambda$$



Proposition 47. *Suppose that $\pi \in \Lambda^p$ and $\sigma \in \Lambda^q$ are the Plücker coordinates of a p-plane and a q-plane in V, intersecting in a subspace with coordinate $I \in \Lambda^k$ and spanning a subspace with coordinate $S \in \Lambda^{p+q-k}$.*

1. $\psi^{p\,q}_{l\,(p+q-l)}(\pi, \sigma) = 0$ *for* $l < k$
2. $\psi^{p\,q}_{k\,(p+q-k)}(\pi, \sigma) \equiv I \otimes S$

Proof. The maps ψ are $GL(V)$ module maps. $GL(V)$ is transitive on pairs consisting of a p-plane and a q-plane intersecting in k dimensions, so it suffices to contrive the conclusion in a special case. Suppose

$$\pi = i_1 i_2 .. i_k \pi_1 .. \pi_{p-k}$$
$$\sigma = i_1 i_2 .. i_k \sigma_1 .. \sigma_{q-k}$$

In the expression of $\Delta(\pi)$, any terms with right-hand i factors will subsequently map to zero under the last map comprising ψ (σ contains a copy of each such factor, and the exterior product of identical i's is zero). Thus we need only consider those terms of $\Delta(\pi)$ in which all i factors appear on the left. In particular, there are no terms in the end with fewer than k left-hand factors; this proves (1).
On the other hand, there is precisely one such term in which there are k left-hand factors and all k of the $i_1..i_k$ appear. The application of the last map comprising ψ upon this term is evidently equal to

$$\pm i_1..i_k \otimes \pi_1..\pi_{p-k} i_1..i_k \sigma_1..\sigma_{q-k} \equiv I \otimes S$$

This proves (2).

□

Note 48. (Specialization to exterior product and meet). The exterior product is the map $\psi^{p\,q}_{0\,p+q}$ with respect to the evident isomorphism $\Lambda^0 \otimes \Lambda^{p+q} \cong \Lambda^{p+q}$, and the "meet" is the map $\psi^{p\,q}_{(p+q-n)\,n}$ with respect to the isomorphism $\Lambda^{(p+q-n)} \otimes \Lambda^n \cong \Lambda^{(p+q-n)}$ determined by a volume form in Λ^n.

Note 49. (Generalization to several planes). Tensor products of the map ψ with itself and the identity can be composed with themselves to calculate: intersections and spans of any number of planes, the intersections and spans of *these*, and so on.

3.2 Calculating the edge of regression

In the setting of the previous section, section 3.1, consider the case $\dim V = 4$.

Proposition 50. *Suppose that $l(t) \in \Lambda^2 V$ describes a ruled surface in $P^3 := \mathbb{P}V$, meaning that $l(t)$ is a smooth function of the real parameter t and each $l(t) \neq 0$ is decomposable.*

1. *The ruled surface $l(t)$ is developable if and only if $l'(t) \in \Lambda^2$ is also decomposable.*
2. *If $l(t)$ is developable and not a cone, the factors of*

$$\psi^{2\,2}_{1\,3}(l(t), l'(t)) \equiv I(t) \otimes S(t)$$

are representatives $I(t) \in \Lambda^1$ of the edge of regression of the developable surface and $S(t) \in \Lambda^3$ of its osculating plane.

Proof. (1) See [Hla53].
(2) The stipulation that $l(t)$ describes a developable surface which is not a cone means that there is a smooth function $z(t) \in V$, with $z'(t)$ linearly independent from $z(t)$, representing a curve in $\mathbb{P}V$ which is the edge of regression of the ruled surface $l(t)$. The tangent line of this curve has Plücker coordinate $l(t)$ by definition, but this coordinate is also proportional to

$$z(t) \wedge z'(t)$$

Let $y(t) = f(t)z(t)$, where $f(t) \in \mathbb{R}$ is a function such that $z(t) \wedge z'(t) f(t)^2 = l(t)$ (replacing $l(t)$ with $-l(t)$ if necessary). Then $y(t)$ also represents the edge of regression, and

$$l(t) = y(t) \wedge y'(t)$$

Note that the osculating plane of the curve represented by $y(t) \equiv z(t)$ has coordinate $y(t) \wedge y'(t) \wedge y''(t)$.
Differentiating,

$$l'(t) = y'(t) \wedge y'(t) + y(t) \wedge y''(t) = y(t) \wedge y''(t)$$

According to Proposition 47,

$$\psi_{1\,3}^{2\,2}(l(t), l'(t)) \equiv I(t) \otimes S(t) \equiv y(t) \otimes y(t) \wedge y'(t) \wedge y''(t)$$

Thus $I(t)$ represents the edge of regression, and $S(t)$ represents the osculating plane. ☐

3.2.1 Alternative abstract description of the linear map $\psi_{1\,3}^{2\,2}$

The map $\psi_{1\,3}^{2\,2}$ turns out to be anti-symmetric in its two inputs $\Lambda^2 V$. That is, it descends to a linear map

$$\Lambda^2(\Lambda^2 V) \to \Lambda^1 V \otimes \Lambda^3 V \cong \operatorname{End} V \tag{3.1}$$

The $SL(V)$ representation $\Lambda^2(\Lambda^2 V)$ is irreducible of dimension 15; it turns out to be isomorphic to the adjoint representation $\mathfrak{sl}(V)$. Therefore by Schur's lemma it admits an $SL(V)$-isomorphism onto $\mathfrak{sl}(V) \subset \operatorname{End}(V)$, unique up to scalar multiplication.

In Proposition 50, for each time t there was a pair of elements $l, l' \in \Lambda^2 V$ satisfying

1. $l \wedge l = 0$

   (l is decomposable; l is a representative of an element of the Grassmannian $\operatorname{Gr}(2,4) \subset \mathbb{P}\Lambda^2 V$)

2. $l \wedge l' = 0$

   (l' is a representative of a tangent vector to the Grassmannian at $[l]$)

3. $l' \wedge l' = 0$

   ((l, l') represents a null vector in $T_l \operatorname{Gr}(2,4)$, as defined in Example 45)

Note that the conditions (1), (2), (3) on (l, l') are equivalent to the condition that the span of l, l' is a subspace of Λ^2 which is isotropic for the quadratic form equal to the exterior square.

An argument by $GL(V)$ equivariance of the map (3.1) and orbit-matching shows that these conditions are equivalent to the condition that the corresonding element $m \in \mathfrak{sl}(V) \subset \operatorname{End} V$ has rank 1. That is, even without using Proposition 47, we know that m has the form $I \otimes S$, where $I \in V$ and $S \in V^*$ (with special linear condition $S(I) = 0$).

3.2.2 Alternative concrete description of $\psi^{2\,2}_{1\,3}$

As an $SL(V)$ representation, $\Lambda^2 V$ is isomorphic to its dual, and $\Lambda^1 V$ and $\Lambda^3 V$ are dual, so that $\psi^{2\,2}_{1\,3}$ can be regarded as an $SL(V)$ invariant linear function on the representation

$$W := \Lambda^2 V \otimes \Lambda^2 V \otimes \Lambda^1 V \otimes \Lambda^3 V$$

Grosshans, Rota, and Stein [GRS87] explain in detail how to calculate all $SL(V)$ invariant polynomial functions of such tensor products, by means of the *umbral calculus*. They are represented abstractly by elements of an algebra known as the *bracket algebra*. The corresponding expression with respect to a basis for V is obtained by applying an operator known as the *umbral operator*.
I implemented this calculus in the computer algebra system Macaulay2. In the case of W, the so-called linear monomial generators of the bracket algebra over the integers are given by 4-fold bracket expressions in symbols a, b, c, d, labelling the factors of W, in which a appears 2 times, b appears 2 times, c appears 1 time, and d appears 3 times. More precisely, in this case there are 8 linear bracket monomials,

1. $[aabb][cddd]$
2. $[abcd][abdd]$
3. $[bbcd][aadd]$
4. $[aacd][bbdd]$
5. $[aabc][bddd]$
6. $[bbac][addd]$
7. $[bbad][acdd]$
8. $[aabd][bcdd]$

belonging to a rank 2 lattice in the 2-dimensional space of linear invariants $(W^*)^{SL(V)}$ as shown:

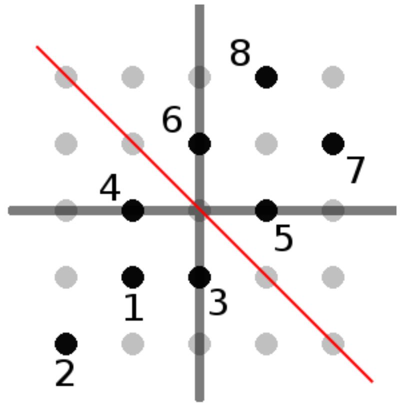


Figure 3: Umbral images of the bracket monomials in $(\Lambda^2 \otimes \Lambda^2 \otimes \Lambda^1 \otimes \Lambda^3)^{*SL(4)}$

The elements proportional to $\psi^{2\,2}_{1\,3}$ form one main diagonal, shown in red. The other main diagonal contains the umbral images of the monomials $[aabb][cddd]$ and $[abcd][abdd]$, which are proportional to the more obvious invariant element given by the exterior product of the factors $\Lambda^2 \otimes \Lambda^2$ times the exterior product of the factors $\Lambda^1 \otimes \Lambda^3$.
This provides several umbral representations of $\psi^{2\,2}_{1\,3}$, for example:

$$\begin{aligned} &[bbcd][aadd] + [aabc][bddd] \\ \equiv&[aacd][bbdd] + [bbac][addd] \\ \equiv&[bbcd][aadd] - [aacd][bbdd] \\ \equiv&[aabc][bddd] - [bbac][addd] \end{aligned}$$

By appying the umbral operator with the Macaulay2 program, we easily calculate the formula for $\psi^{2\,2}_{1\,3}$ with respect to a basis for V. Let $a_{ij}, b_{ij}, c_i, d_{ijk}$ denote tensor coordinate functions on the respective factors of W. The formula is given by the product of the following 3 matrices:
The first matrix is

$$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

The second matrix is

$$\begin{bmatrix} \delta_1 & -2a_{42}b_{32}+2a_{32}b_{42} & -2a_{43}b_{32}+2a_{32}b_{43} & -2a_{43}b_{42}+2a_{42}b_{43} \\ 2a_{41}b_{31}-2a_{31}b_{41} & \delta_2 & 2a_{43}b_{31}-2a_{31}b_{43} & 2a_{43}b_{41}-2a_{41}b_{43} \\ -2a_{41}b_{21}+2a_{21}b_{41} & -2a_{42}b_{21}+2a_{21}b_{42} & \delta_3 & -2a_{42}b_{41}+2a_{41}b_{42} \\ 2a_{31}b_{21}-2a_{21}b_{31} & 2a_{32}b_{21}-2a_{21}b_{32} & 2a_{32}b_{31}-2a_{31}b_{32} & \delta_4 \end{bmatrix},$$

where for typographical reasons we show the diagonal entries separately:

$$\begin{aligned}
\delta_1 &= a_{43}b_{21} - a_{42}b_{31} - a_{41}b_{32} + a_{32}b_{41} + a_{31}b_{42} - a_{21}b_{43} \\
\delta_2 &= a_{43}b_{21} + a_{42}b_{31} + a_{41}b_{32} - a_{32}b_{41} - a_{31}b_{42} - a_{21}b_{43} \\
\delta_3 &= -a_{43}b_{21} - a_{42}b_{31} + a_{41}b_{32} - a_{32}b_{41} + a_{31}b_{42} + a_{21}b_{43} \\
\delta_4 &= -a_{43}b_{21} + a_{42}b_{31} - a_{41}b_{32} + a_{32}b_{41} - a_{31}b_{42} + a_{21}b_{43}
\end{aligned}$$

This third matrix is:

$$\begin{bmatrix} d_{432} \\ -d_{431} \\ d_{421} \\ -d_{321} \end{bmatrix}$$

3.2.3 Topological-combinatorial interpretation of the bracket algebra

For simple combinatorial reasons, the bracket monomials representing linear invariants of $W := \Lambda^{p_1}V \otimes ... \otimes \Lambda^{p_k}V$ can be identified with certain simplicial complexes, and the additive relations they satisfy can be expressed as certain "surgery rules". One such rule is depicted in Figure 4.

Let $\dim V = n$. Linear invariants of W only exist if n divides $p_1 + ... + p_k$. Let m be the quotient integer in this case. Each linear bracket monomial is a product of m n-fold brackets, in which a symbol labelling $\Lambda^{p_1}V$ appears p_1 times, a symbol labelling Λ^{p_2} appears p_2 times, etc. Since the bracket expressions commute, each linear bracket monomial can be identified with a unique regular simplicial complex with m disjoint unlabelled n-simplices and one labelled (colored) $(p_i - 1)$-simplex for each factor Λ^{p_i}, in which every vertex of an n-simplex belongs to exactly one of the $(p_i - 1)$-simplices.

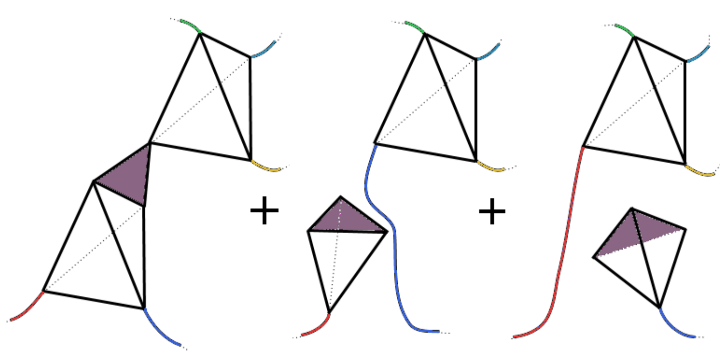


Figure 4: A relation (equal to 0) in the bracket algebra corresponding to invariant functions on a tensor product of $SL(4)$ representations of the form $... \otimes \Lambda^3 \otimes ...$.

3.3 Focal surfaces of line congruences

The calculation of the edge of regression of a developable surface in Proposition 50 can be readily extended to a calculation of a pair of surfaces known as the focal surfaces associated to a 2-parameter family of lines in projective 3-space.

We regard the family, classically called a congruence, as a smooth immersed surface S in $\mathrm{Gr}(2,4)$. There is an open subset $S_o \subset S$ consisting of points l at which the projective line $\mathbb{P}T_l S$ meets the null quadric $N_l \subset \mathbb{P}T_l\,\mathrm{Gr}(2,4)$ in two distinct points (see Example 45 for the definition of N_l). These two points are the data of a locally-defined pair of transverse foliations on S_o, also known as a 2-web or net, the *focal net*. Locally, they organize the 2-parameter family of lines S_o as a 1-parameter family of developable surfaces, in two ways. The union of the edges of regression of each family is called a *focal surface* of S.

In the general language of section 2: The total space of the family E has dimension 3, the base $B = S$ has dimension 2, and $X = P^3$. For a line $l \in S$, the nullity 1 characteristic generically consists of two points of l. The stationary family of 0-planes in $\mathbb{P}T_l S \cong \mathbb{RP}^1$ consists of the two null directions, comprising the focal net in S. The focal surface of S is its envelope of nullity 1.

Question 51. *Does the calculation of order $s = 1, 2$ and nullity $n = 1$ envelopes of 2-plane families and focal surfaces of line families in $\mathbb{P}^3$, afforded by the linear map $\psi^{2\,2}_{1\,3}$, extend to a calculation of the envelopes of nullity n and order s of r-parameter families of m-planes in $\mathbb{P}^n$ by means of the maps $\psi^{p\,q}_{k\,(p+q-k)}$?*

3.3.1 Examples

Example 52. (The Backlünd transformation). A smooth family S of tangent lines to each point of surface M in 3-dimensional projective space is always itself one of the focal surfaces of the congruence S.

Backlünd showed in 1880 that when $M \subset \mathbb{R}^3$ is a surface of constant negative curvature with respect to a choice of Euclidean structure on the ambient space, there is a congruence S of tangent lines to M whose second focal surface *also* has constant negative curvature. See [Cod05] for a comprehensive exposition of this transform and other properties of the so-called pseudospherical surfaces.

Example 53. (Surface evolutes). The two focal surfaces of the congruence of

normal lines to a surface $M \subset \mathbb{R}^3$ are called the *evolutes*, or the *caustic*[1] of the surface. In this case the focal net is the two transverse families of lines of curvature of M. This can be regarded as a synthetic way to define the lines of curvature.
See e.g. [Dar72] (originally published 1887).

Example 54. (Kummer surface). The original Kummer surface [Kum64] is a quartic hypersurface in projective 3-space which is the focal surface of certain line congruences belonging to the so-called quadratic line complex, the 3-fold equal to the zero set of a section $s \in H^0(\mathrm{Gr}(2,4), \mathcal{O}(2))$ ([Jes03], page 150).

Example 55. (Geodesic flow on the ellipsoid). Consider a non-spherical ellipsoid $M \subset \mathbb{R}^3 \subset \mathbb{P}(\mathbb{R}^4)$, given as the zero set of quadratic form Q diagonalized with respect to the standard basis of $\mathbb{R}^4$:

$$Q(x) = \frac{x_1^2}{q_1} + \frac{x_2^2}{q_2} + \frac{x_3^2}{q_3} - x_4^2$$

The classical term *confocal quadric* refers to one of the members Q_λ of the family described by the formula

$$Q_\lambda(x) = \frac{x_1^2}{q_1 + \lambda} + \frac{x_2^2}{q_2 + \lambda} + \frac{x_3^2}{q_3 + \lambda} - x_4^2 \quad \lambda \in \mathbb{R}$$

Jacobi showed in [Jac39] that if γ is a geodesic in M with respect to the Riemannian metric it inherits as a subspace of Euclidean 3-space, there is a $\lambda \in \mathbb{R}$ such that every tangent line of γ is also tangent to the confocal quadric Q_λ. This was used to prove the Liouville-Arnold integrability of the geodesic flow in TM over M. The set of all tangents lines of geodesics γ in M tangent to a fixed Q_λ form a congruence, whose two focal surfaces are M and Q_λ. One of the foliations comprising its focal net consists of orbits for the geodesic flow in $\mathbb{P}TM$.
Note that, due to integrability, the base of one of these congruences almost always forms a closed surface of genus 1.

3.3.2 Confocal quadrics

The classical notion of confocal quadrics in Example 55 can be abstracted as follows.

[1]From the Latin *causticus*, meaning "burning", as in the consequence of convergence of rays of light. Although the word *focus* is sometimes regarded with the same sense, as in *focal surface*, the Latin word *focus* actually means "hearth" or "fireplace", with emphasis on the center of attention rather than a literal place of burning.

Definition 56. *Let k be a field. A* generalized n-dimensional (pseudo-) Euclidean space over k *is a triple* $(V, W, [q])$*:*

1. *A k-vector space V of dimension $n+1$*
2. *A vector subspace $W \subset V$*
3. *A non-zero scalar equivalence class of non-degenerate symmetric tensor $[q] \in \mathbb{P}S^2W$*

The underlying space *is the image of $V \backslash W$ in $\mathbb{P}V$. Its group of* isometries *is*

$$\mathrm{Euc}(V, W, [q]) = \left\{g \in GL(V) | gW \subset W, g|_W \in O(q, W), g_{V/W} = \mathrm{id}\right\}$$

Although the other cases may also be interesting, in the rest of this section we shall assume that the field k is $\mathbb{R}$ or $\mathbb{C}$ and that the codimension of W in V is 1.

Note 57. Suppose $k = \mathbb{R}$. If q has definite signature, then $(V, W, [q])$ describes the projective model for an ordinary Euclidean space of dimension n. A flat Riemannian metric on the underlying space is uniquely determined up to a uniform scalar. If q has indefinite signature, then $(V, W, [q])$ describes the projective model for a generalized Minkowski space, or contractible flat pseudo-Riemannian manifold.

Recall that the *dual* of a non-degenerate quadratic form $A \in S^2V^*$ is the tensor $a := \langle A, \tilde{A}^{-1} \otimes \tilde{A}^{-1}\rangle \in S^2V$, where

1. $\tilde{A}$ denotes the linear isomorphism $V \to V^*$ given by $v \mapsto A(v, -)$, and
2. $\langle , \rangle$ denotes composition or tensor contraction.

Remark 58. With respect to a basis for V and the attendant dual basis for V^*, the matrix of A as a bilinear form and the matrix of the dual a as a symmetric tensor are related by matrix inversion.

Definition 59. *Let $E = (V, W, [q])$ be a Euclidean structure, and let $A, B \in S^2V^*$ denote generators for the ideals of two non-degenerate quadrics in $\mathbb{P}V$. Let $a, b \in S^2V$ denote their duals. The quadrics shall be called* confocal with respect to E *if a, b, and q lie on a pencil.*

Note 60. Every quadratic form on a finite-dimensional $\mathbb{R}$ or $\mathbb{C}$ vector space is diagonalizable. In fact, every pair of such quadratic forms A, B is simultaneously diagonalizable. The most familiar case of this is the fact that a real symmetric matrix can be orthogonally diagonalized; in that case A is the

standard Euclidean inner product, and B is the quadratic form represented by the symmetric matrix.
This implies that all members of a pencil of quadratic forms are simultaneously diagonalizable. In particular, if A and B are confocal with respect to $(V, W, [q])$, then a, b, and q are simultaneously diagonalizable.

Note 61. (Specialization to classical confocality). Let us see how Definition 59 generalizes the notion described in Example 55.
Suppose that given non-degenerate $A, B \in S^2V^*$ are confocal with respect to a fixed 3-dimensional (real) Euclidean structure $(V, W, [q])$. Regarding a, b, and $q \in S^2W \subset S^2V$ as quadratic forms on V^*, find a basis for V^* with respect to which a, b, q are simultaneously diagonalized. This basis must extend a basis l for the 1-dimensional kernel of q in V^*, which is also equal to $W^\perp$. Assume that the last basis element is l.
Let us use the notation (t) for the matrix of a tensor t.
After rescaling the first 3 basis elements, we can assume that the matrices with respect to this basis are of the form

$$\begin{aligned}(a) &= \mathrm{diag}(a_1, a_2, a_3, a_4)\\ (b) &= \mathrm{diag}(b_1, b_2, b_3, b_4)\\ (q) &= \mathrm{diag}(1, 1, 1, 0)\end{aligned} \tag{3.2}$$

By definition there are scalars $\mu, \nu, \lambda \in \mathbb{R}$ such that $\mu(a) = \nu(b) + \lambda(q)$. By rescaling a and b, we can assume that $\mu = \nu = 1$ and the matrices still have the form 3.2. Only now $a_4 = b_4$, so we can scale the last basis element (and, if necessary, rescale a and b by -1) to assume that $a_4 = b_4 = -1$:

$$\begin{aligned}(a) &= \mathrm{diag}(a_1, a_2, a_3, -1)\\ (b) &= \mathrm{diag}(b_1, b_2, b_3, -1)\\ (q) &= \mathrm{diag}(1, 1, 1, 0)\end{aligned}$$

According to Remark 58,

$$\begin{aligned}(A) &= \mathrm{diag}(a_1^{-1}, a_2^{-1}, a_3^{-1}, -1)\\ (B) &= \mathrm{diag}(b_1^{-1}, b_2^{-1}, b_3^{-1}, -1)\end{aligned}$$

The condition $(a) = (b) + \lambda(q)$ is

$$a_i = b_i + \lambda, \qquad i = 1, 2, 3$$

which is equivalent to $A = B_\lambda$ in the notation of Example 55. So A and B are confocal in the classical sense, when regarded as diagonalized quadrics in ordinary Euclidean space.

Note 62. Let $A, B \in S^2V^*$ be non-degenerate. We'll show that A and B are very often confocal with respect to some Euclidean or pseudo-Euclidean structure $(V, W, [q])$.
Denote the duals of A, B by $a, b \in S^2V$. Recall that $n := \dim V$. Let $D \subset \mathbb{P}(S^2V)$ denote the degree n determinantal hypersurface, and consider the pencil P spanned by $[a]$ and $[b]$ in $\mathbb{P}(S^2V)$.
In the complexification, P meets D in n points, accounting for multiplicity. Since we are working over $\mathbb{R}$, this number can be less, but the typical situation is still for P to meet D in its highest-dimensional stratum, the smooth locus consisting of quadratic forms on V^* of rank $n-1$.
Suppose that q is such a form. Then $K := \ker q \subset V^*$ has dimension 1. In this case set $W := K^\perp \subset V$. By definition of K, q descends to a non-degenerate form on V^*/K. On the other hand V^*/K is naturally isomorphic to W^* by the non-degenerate pairing

$$(V^*/K) \otimes K^\perp \to \mathbb{R}$$

Thus q can be regarded as a non-degenerate quadratic form on W^*, and hence a non-degenerate element of S^2W. Therefore A and B are confocal with respect to the pseudo-Euclidean structure $(V, W, [q])$.

Note 63. The reasoning in Note 62 also has the following geometric consequence. The Euclidean or pseudo-Euclidean structures on V are identified with the minimally singular quadrics in $\mathbb{P}V^*$. That is, the quadrics with an isolated singular point.
It is tricky to "see" this structure over $\mathbb{R}$ in the case of ordinary Euclidean structures; the quadric in $\mathbb{P}V^*$ in this case has only one real point!

Example 64. Let A be an ellipsoid in the 3-dimensional real projective space. According to Note 62, for many choices of second quadric B, the congruence of lines tangent to A and to B can be realized as the tangent lines to a foliation of A by geodesics for an appropriately chosen metric on A.

3.3.3 Application to visual surface reconstruction

In this section we sacrifice precision in favor of brevity and clarity by employing the terminology of visual geometry.
Suppose that an object in $\mathbb{R}^3 \subset \mathbb{RP}^3$ is described by a smooth surface X. We study X from a 1-parameter family of observation points, varying with time. The generic local behavior is for the silhouette of the surface at each time to be bounded by a smooth curve in the observer's retinal image plane. If the observer is aware of its spatial position and orientation, the data of

these smooth curves can be organized into a smooth surface $S \subset \mathrm{Gr}(2,4)$, the 2-parameter family of lines which are "lines of sight" tangent to X and passing through one of the observation points. The 2 parameters are time and the silhouette boundary arc.

Fix $l \in S$. With respect to any chosen basis of $T_l S$, the two null directions in $T_l S$ are determined by the standard quadratic formula. One solution corresponds to a part of the conical developable surface, for fixed time, consisting of all lines of sight tangent to X and passing through a fixed position of the observer. In this case the edge of regression of the developable degenerates to a point, the fixed observer position.

The edges of regression of the developables of the other, transverse family lie upon and locally foliate X. The calculation of Proposition 50 can be regarded as a calculation of the points of the object X purely in terms of the visual observations S.

Note 65. I obtained the idea to undertake such a calculation directly from the book of Cipolla and Giblin [CG00]. They explain on page 86 how to solve this exact same problem in the Euclidean setting as a calculus exercise, by finding a formula for the distance between the observer and the points of the object X seen as boundary points of its silhouette. The alternative described in this section has the advantage of being well-defined in the broader projective geometry.

Incidentally, the distance formula of [CG00] turns out to be an independently obtained special case of the formula of Ramm and Katsevich for the singular support of a function in terms of its generalized Radon transform ([RK96], pages 123-124).

Note 66. (*Half-geodesics*). Let us call the congruences S arising from this visual-geometric construction *visual congruences*. Visual congruences have the property that the focal surface of one of the developable families degenerates to a curve, namely the path of the observer. Hlavaty showed that one of the null foliations of such congruences satisfies a system of first-order differential equations ([Hla53] page 151). We explain another interpretation of this property of visual congruences.

Recall that the tangent bundle of $\mathrm{Gr}(2,V)$ can be identified with the tensor product $\tau^* \otimes (V/\tau)$, where τ denotes the tautological rank 2 vector bundle. The tensor factors are also known as spin bundles (see [WW90]). With respect to this description, the null directions N are precisely the rank 1 tensors, the elements belonging to the determinant locus. Let the null direction fields tangent to a visual congruence S be expressed by

$$n_1 = p_1 \otimes q_1 \qquad n_2 = p_2 \otimes q_2,$$

One of them, say n_1, has the property that p_1 is constant along n_1. Here "constant" can be taken to mean constant with respect to the canonically-defined embedding of τ in the trivial vector bundle with fiber V, in the sense that there is a fixed element $f \in V^*$ whose images in τ^* over an integral curve of n_1 are proportional to p_1. Alternatively, p_1 is covariant constant in the direction n_1, i.e. $\nabla_{n_1} p_1 = 0$, where ∇ denotes a connection on the spin bundles compatible with the Levi-Civita connection of a pseudo-Riemannian metric on $\mathrm{Gr}(2,4)$ vanishing on the null directions N.
This is precisely the condition that the curves tangent to n_1 are "half-geodesics", in the language of ([BLN15], page 5). There, this condition is used to define a special system of null curves on a different signature $(2,2)$ conformal pseudo-Riemannian 4-manifold M, not our $\mathrm{Gr}(2,4)$. This system is more complicated than the one we use over $\mathrm{Gr}(2,4)$, because $\mathrm{Gr}(2,4)$ is conformally flat, while M is not. Their system is shown to consist of the images of the integral curves to a certain 2-dimensional distribution in a 5-manifold Q fibering over M, whose symmetry algebra is the exceptional Lie algebra $\mathfrak{g}_2$.

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