Multi-focal tensors as invariant differential forms & Envelopes of line congruences for Structure-From-Motion

SIAM mini-symposium in Algebraic Vision

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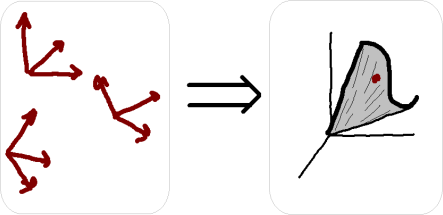
Idea of multi-focal tensors



e.g. $t \in (\mathbb{R}^3)^{\otimes 3}$

- Equations of the variety?
- (Linear) constraints on a given point from measurements of the frame?
- Inversion?
- Why do these structures exist?

V	f.d. vector space over k
$\mathbb{P} :=$	$\mathbb{P}V \cong GL(V)/H$
$[v_0] \in \mathbb{P}$	basepoint
$T_0 :=$	$T_{[v_0]}$ tangent space
$\Lambda^p :=$	$\Lambda^p V^*$
$S^q :=$	$S^q V^*$
$F :=$	$\text{Frames}(\mathbb{P}V) \cong GL(V)/K$

Skew-tensors are Koszul cycles are differential forms

Euler sequence,
p^{th} exterior power,
dual,
$p + q$ twist,
global sections

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$$0 \to k \to V(1) \to T_{\mathbb{P}} \to 0$$

$$0 \to \dots \to \dots \to \dots \to 0$$

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$$0 \to H^0(\Omega^p(p+q)) \to \Lambda^p \otimes S^q \xrightarrow{\delta} \Lambda^{p-1} \otimes S^{q+1}$$

Skew-tensors are functions of a frame

What we win:

$\Lambda^{p+1} \cong H^0(\Omega^p(p+1))$ (Lie 1877)

Skew-tensors are functions of a frame

What we win:

$$\Lambda^{p+1} \cong H^0(\Omega^p(p+1)) \qquad \text{(Lie 1877)}$$

$$\cong \left\{ GL(V) \overset{H\,\text{equiv.}}{\rightarrow} \Lambda^p T_0(\otimes(\text{1-dim})) \right\}$$

$$\subset \left\{ GL(V) \overset{K\,\text{equiv./inv.}}{\rightarrow} \Lambda^p T_0 \right\}$$

$$\cong \{F \to \Lambda^p T_0\}$$

Multi-focal varieties

Theorem (Construction)

For each $GL(V)$ invariant

$$I \in \Lambda^{p_1+1} \otimes \cdots \otimes \Lambda^{p_n+1}$$

there is a corresponding $GL(V)$ invariant (rational) function

$$f(I) : F^n \to \Lambda^{p_1} T_0^* \otimes \cdots \otimes \Lambda^{p_n} T_0^*$$

$f(I)$	multi-focal map
image$(f(I))$	multi-focal variety

Examples of multi-focal varieties ($\dim V = 4$, $k = \mathbb{R}$)

- $I_2 \in \Lambda^2 \otimes \Lambda^2 \rightsquigarrow$ bifocal tensors in $\mathbb{R}^3 \otimes \mathbb{R}^3$ (fundamental matrices / essential matrices) Longuet-Higgins, Faugeras, Demazure, Mourrain, Luong
- $I_3 \in \Lambda^3 \otimes \Lambda^2 \otimes \Lambda^3 \rightsquigarrow$ trifocal tensors in $(\mathbb{R}^3)^{\otimes 3}$ Hartley, Zisserman, Faugeras, Luong, Alzati, Tortora, Papadopoulo, Aholt, Oeding
- $I_4 \in (\Lambda^3)^{\otimes 4} \ldots$
- $I_6 \in (\Lambda^2)^{\otimes 6}$? (Line complex)

Euclidean trifocal variety

$$r, s \in O(3) \subset \mathbb{R}^3 \otimes \mathbb{R}^3,\ u, v \in \mathbb{R}^3$$
$$(r, u), (s, v) \mapsto t = -r \otimes v + u \otimes s$$

$t = (t_1, t_2, t_3), \quad t_i \in \mathbb{R}^3 \otimes \mathbb{R}^3$
$a_i :=$ adjugate t_i

Theorem

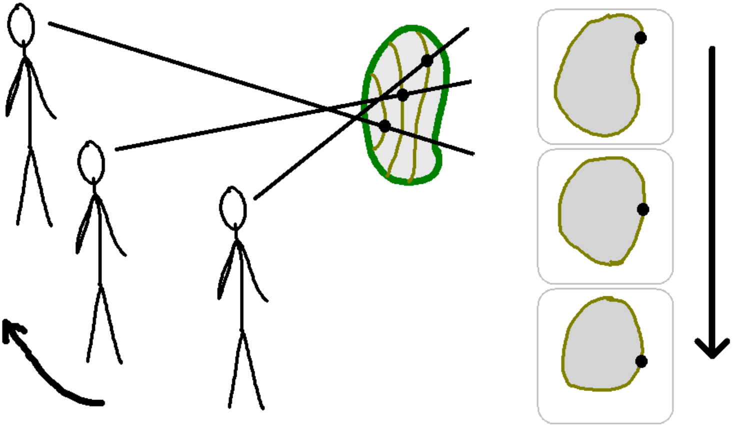
If $t \in (\mathbb{R}^3)^{\otimes 3}$ is a Euclidean trifocal tensor,

$$t_i a_j t_i = t_j a_i t_j \qquad \forall i \neq j$$
$$a_i t_j a_k = 0 \qquad \forall \text{distinct } i, j, k$$

The equations of the Euclidean trifocal variety are not all known.

And now for something completely different

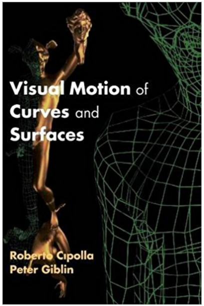
Contours in multiple views



Contours in multiple views

Koenderink 1984
What does the occluding contour tell us about solid shape?

Cipolla and Giblin 1994
Visual motion of curves and surfaces.



distance formula
Do we need the Euclidean geometry?

Focal envelope

S surface $\subset \text{Gr}(2,4) =$ lines in $\mathbb{P}^3$
$(2,2)$ conformal metric on $\text{Gr}(2,4)$
$(1,1)$ conformal metric on S
two null foliations of S
leaf $\leftrightarrow$ developable surface

Jessop 1903
Eisenhardt 1909
Lane 1932
Hlavaty 1953

Osculating flag of null tangent vector in $\mathrm{Gr}(2,4)$

$X = X(t) \in \Lambda^2 V$ developable surface:

$$X \wedge X = 0 \qquad X' \wedge X = 0 \qquad X' \wedge X' = 0$$

$P = P(t) \in \Lambda^1 V = V$ edge of regression of X
$\Pi = \Pi(t) \in \Lambda^3 V \cong V^*$ tangent planes of X

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Theorem

$$\psi(X \widetilde{\wedge} X') \equiv P \otimes \Pi$$

where
$\psi : \Lambda^2(\Lambda^2 V) \cong \mathfrak{so}(3,3) \cong \mathfrak{sl}(4) \subset \mathfrak{gl}(4) \cong V \otimes V^*$

Implementation of Structure-From-Motion algorithm

Joint with Paul Xin Zhou (PhD U. Mich.—Google—Baidu)

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